

DISTINCTION OF THE STEINBERG REPRESENTATION III: THE TAMELY RAMIFIED CASE

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ABSTRACT. Let F be a nonarchimedean local field, let E be a Galois quadratic extension of F and let G be a quasisplit group defined over F ; a conjecture by Dipendra Prasad states that the Steinberg representation St_E of $G(E)$ is then χ -distinguished for a given unique character χ of $G(F)$, and that χ occurs with multiplicity 1 in the restriction of St_E to $G(F)$. In the first two papers of the series, Broussous and the author have proved the Prasad conjecture when G is F -split and E/F is unramified; this paper deals with the tamely ramified case, still with G F -split.

1. INTRODUCTION

Let F be a nonarchimedean local field with finite residual field, let E be a Galois quadratic extension of F and let G be a reductive group defined over F . Let G_E (resp. G_F) be the group of E -points (resp. F -points) of G and let π be a smooth representation of G_E ; we say that π is *distinguished* with respect to the symmetric space G_E/G_F if the space $\text{Hom}_{G_F}(\pi, 1)$, where 1 is the one-dimensional trivial representation of G_F , is nontrivial. This article deals with the important particular case of the distinction of the Steinberg representation of G_E .

In [16], Dipendra Prasad has proved that when $G = GL_2$, the Steinberg representation St_E of G_E is not distinguished with respect to G_E/G_F ; on the other hand, if we set $\chi = \varepsilon_{E/F} \circ \det$, where $\varepsilon_{E/F}$ is the norm character of E^*/F^* , the space $\text{Hom}_{G_F}(St_E, \chi)$ happens to be of dimension 1. For that reason, the definition of distinguishedness will be extended the following way: let χ be any character of G_F ; we say that π is χ -*distinguished* with respect to G_E/G_F if $\text{Hom}_{G_F}(\pi, \chi)$ is nontrivial.

In [17], Prasad has stated a conjecture about the distinction of the Steinberg representation which generalizes his result of [16]; the conjecture, as initially stated, concerns quasisplit groups, but can be extended to any connected reductive group (see [18]). Let G^{ad} be the adjoint group G/Z , where Z is the center of G , and let χ_{ad} be some given character of the group G_F^{ad} of F -points of G^{ad} , called the Prasad character (see [17] for the definition of the Prasad character in the case of a F -quasisplit group, and [18] for its extension to the general case).

Note that since this article only deals with F -split groups, we will simply use [17] as a reference for its definition.) We then have:

Conjecture 1 (Prasad). *The representation St_E is χ_{ad} -distinguished with respect to $G^{ad}(F)$, and $\text{Hom}_{G^{ad}(F)}(St_E, \chi_{ad})$ is one-dimensional. Moreover, St_E is not χ' -distinguished for any character χ' of $G^{ad}(F)$ distinct from χ_{ad} .*

It is not hard to see that the above conjecture is equivalent to the same one with G_{ad} replaced with G and χ_{ad} with the Prasad character χ of G_F . The result has been proved for $G = GL_n$ and F of characteristic 0 by Anandavardhanan and Rajan ([1]), and more recently by Matringe for G being an inner form of GL_n and F of characteristic different from 2 ([15]). It has also been proved for any F -split G by Broussous and the author ([5] and [10]) when E/F is unramified; the present article deals with the tamely ramified case. More precisely, we prove the following results, which are the respective analogues of [5, theorems 1 and 2]: let χ be the Prasad character of G_F relative to E/F ; we have:

Theorem 1.1. *Assume G is split over F and E/F is totally and tamely ramified. The Steinberg representation St_E of G_E is then χ -distinguished with respect to G_F .*

Theorem 1.2. *With the same hypotheses, the character χ occurs with multiplicity at most 1 in the restriction to G_F of St_E , and St_E is not χ' -distinguished for any character χ' of G_F distinct from χ .*

By the previous remarks we do not lose any generality by assuming that G is semisimple and adjoint. To make proofs clearer, we even assume that G is simple, the general case of semisimple groups being easy to deduce from the simple case.

The proof uses the model of the Steinberg representation that was already used in [5]: the Steinberg representation can be viewed as the space of smooth harmonic cochains over the set of chambers of the Bruhat-Tits building of G_E , with G_E acting on it via its natural action twisted by a character ε (defined in section 3), whose restriction to G_F happens to be trivial when E/F is ramified (proposition 3.2). To prove theorem 1.1, we thus only need to exhibit a (G_F, χ) -equivariant linear form on that space, as well as a test vector for that form. This is done in subsections 7.2 and 7.3. We prove the convergence of our linear form and the existence of a test vector with the help of the Poincaré series of affine Weyl groups (see [14, section 3]), which allows us to get rid of the condition on q we had to impose in [5]: the trick should work in the unramified case as well, which would lead to a simpler proof than the one given in [5] and [10]. The author thanks Paul Broussous and Dipendra Prasad for suggesting him to use these series.

To prove theorem 1.2, as in [5, section 6], we prove the equivalent result that the space of $G_{F,der}$ -invariant harmonic cochains on the building X_E , where $G_{F,der}$ is the derived group of G_F , is of dimension at most 1 (sections 5 and 6). We

will proceed by induction on the set Ch_E of chambers of X_E , as in [5], but since it turns out that contrary to the unramified case, the support of our harmonic cochains is not the whole set Ch_E , the induction we use here is quite different from the one of [5].

We start by partitioning the set of chambers of X_E into F -anisotropy classes the following way: set $\Gamma = \text{Gal}(E/F)$. For every chamber C , there exists a Γ -stable apartment A of X_E containing C and an E -split F -torus T attached to A (proposition 4.1, see also [12]); A and T are not unique, but the F -anisotropy class of T does not depend on the choice of A (corollary 4.9), and we define the F -anisotropy class of C as that class. Our goal is to prove theorem 1.2 with the help of an induction on these classes.

Contrary to the unramified case, the building X_F of G_F is not a subcomplex of the building X_E of G_E , but if we consider their respective geometric realizations \mathcal{B}_F and \mathcal{B}_E , the former is still the set of Γ -stable points of the latter, at least when E/F is tamely ramified, and we can thus consider the set Ch_\emptyset of chambers of X_E whose geometric realization is contained in \mathcal{B}_F ; that set is obviously G_F -stable, but in the ramified case, it contains more than one G_F -orbit of chambers. We thus first have to prove that the restrictions of our $G_{F,der}$ -invariant harmonic cochains to Ch_\emptyset are entirely determined by their value on some given element of Ch_\emptyset .

It quickly turns out that we have to treat the case of groups of type A_{2n} separately from the other cases. In the case of type A_{2n} , the $G_{F,der}$ -invariant harmonic cochains are identically zero on Ch_\emptyset outside a particular orbit of chambers that we call Ch_c (corollary 5.3). We then use an induction (similar in its basic idea to the one of [5, section 6], but technically quite different) to prove that these harmonic cochains are entirely determined by their constant value on Ch_c , which proves theorem 1.2 in this case (corollary 5.17). In the proof of theorem 1.1 in the case of a q large enough, our linear form λ has its support on Ch_c , and our test vector is the Iwahori-spherical vector ϕ_C relative to some given chamber C in Ch_c ; we also compute explicitly the value of $\lambda(\phi_C)$ (proposition 7.5).

In the case of groups of type other than A_{2n} , the $G_{F,der}$ -invariant harmonic cochains are identically zero on the whole set Ch_\emptyset (corollary 5.3 again). In fact, it turns out that we can prove with our induction that these cochains are identically zero on the whole set Ch_E outside a unique F -anisotropy class Ch_a , on which the induction fails; that class corresponds to the E -split tori of G whose F -anisotropic component is of maximal dimension (corollary 5.16); we thus may use as a starting point for a new induction the subset Ch_a^0 of the elements of Ch_a which contain a Γ -fixed facet of X_E of the greatest possible dimension; we prove in a similar way as in [5, section 6] that the $G_{F,der}$ -invariant harmonic cochains are entirely determined by their values on Ch_a^0 (corollary 5.6), then we check that the space of the restrictions to Ch_a^0 of our $G_{F,der}$ -invariant harmonic cochains is of dimension at most 1 (section 6). That part of the proof is rather technical because Ch_a^0 does not consist of one single $G_{F,der}$ -orbit in general; it is also the

reason why, to prove theorem 1.1, the test vector we choose in section 7.3 is not an Iwahori-spherical vector. (Note that at the end of the paper (corollary 7.33), we prove that an Iwahori-spherical vector attached to some given element of Ch_a^0 works as well, but using it as a test vector in the first place leads to a more complicated proof.)

The model used in this article and the previous ones can probably be used as well for the remaining cases. For groups whose E -rank and F -rank are the same, the induction should work the same way. For groups whose E -rank and F -rank are different, the induction has to be modified to take into account the fact that the apartments of \mathcal{B}_F are now proper affine subspaces of the apartments of \mathcal{B}_E , but the same basic principle still applies.

The author also expects it to be possible to use the same model and a pretty similar proof to prove the Prasad conjecture in the wildly ramified case as well, but in that case, additional technical problems arise. The main two are the following ones: firstly, it is not true anymore that every chamber of X_E is contained in a Γ -stable apartment; that problem can be addressed by considering, for chambers which do not satisfy that condition, Γ -stable parts of apartments instead of whole apartments, but we still need to extend the result of proposition 5.5 to these bad chambers. Secondly, in the tamely ramified case, the geometric realizations of the Γ -fixed subspaces of such apartments are always contained in \mathcal{B}_F ; this is not true anymore when E/F is wildly ramified, which makes dealing with the values of the harmonic cochains on Ch_a^0 even more complicated than it already is in the tamely ramified case.

This paper is organized as follows. In section 2, we define the notations we use throughout the paper. In section 3, we give the definition of the Prasad character χ , and we check that the χ -distinction of the Steinberg representation is equivalent to the χ -distinction of the natural representation of G_E on the space of the smooth harmonic cochains over its Bruhat-Tits building X_E . In section 4, we separate the set of chambers of X_E into F -anisotropy classes. In section 5, we determine the support of the $G_{F,der}$ -invariant harmonic cochains, and we prove theorem 1.2 in the case of a group of type A_{2n} ; for other types, we reduce the problem to a similar assertion over Ch_a^0 . In section 6, we deal with Ch_a^0 and finish the proof of theorem 1.2 for groups of type different from A_{2n} . In section 7, finally, we prove theorem 1.1.

2. NOTATIONS

Let F be a nonarchimedean local field with discrete valuation and finite residual field. Let E be a ramified Galois quadratic extension of F ; E/F is totally ramified, and is tamely ramified if and only if the residual characteristic p of F is odd.

Set $\Gamma = \text{Gal}(E/F)$; we denote by γ its nontrivial element. We denote by $N_{E/F} : x \mapsto x\gamma(x)$ the norm application $x \mapsto x\gamma(x)$ from E to F .

Let \mathcal{O}_F (resp. \mathcal{O}_E) be the ring of integers of F (resp. E), and let \mathfrak{p}_F (resp. \mathfrak{p}_E) be the maximal ideal of \mathcal{O}_F (resp. \mathcal{O}_E). Let $k_F = \mathcal{O}_F/\mathfrak{p}_F$ (resp. $k_E = \mathcal{O}_E/\mathfrak{p}_E$) be the residual field of F (resp. E); since E/F is totally ramified, k_E and k_F are canonically isomorphic. Let $q = q_E = q_F$ be their common cardinality.

Let ϖ_E be a uniformizer of E , and set $\varpi_F = N_{E/F}(\varpi_E)$. Since E/F is totally ramified, ϖ_F is a uniformizer of F .

Let $v = v_F$ be the normalized valuation on F extended to E ; we have $v(F) = \mathbb{Z} \cup \{+\infty\}$ and $v(E) = \frac{1}{2}\mathbb{Z} \cup \{+\infty\}$.

Let G be a connected reductive group defined and split over F . We fix a F -split maximal torus T_0 of G and a Borel subgroup B_0 of G containing T_0 ; B is then F -split too. Let Φ be the root system of G relative to T_0 ; in the sequel we assume Φ is irreducible. Let Φ^+ be the set of positive roots of Φ corresponding to B_0 , let Δ be the set of simple roots of Φ^+ and let α_0 be the highest root of Φ^+ . We also denote by Φ^\vee the set of coroots of G/T_0 , and by W the Weyl group of Φ .

A Levi subgroup M of G is *standard* (relatively to T_0 and B_0) if $T_0 \subset M$ and M is a Levi component of some parabolic subgroup of G containing B_0 . A root subsystem Φ' of Φ is a *Levi subsystem* if it is the root system of some Levi subgroup of G containing T_0 ; Φ' is *standard* if that Levi subgroup is standard, or in other words if Φ' is generated by some subset of Δ .

For every algebraic extension F' of F and every algebraic group L defined over F' , we denote by $L_{F'}$ the group of F' -points of L .

For every algebraic extension F' of F , let $X_{F'}$ be the Bruhat-Tits building of $G_{F'}$: $X_{F'}$ is a simplicial complex whose dimension is, since G is F -split, the semisimple rank d of G . We have a set inclusion $X_F \subset X_E$ compatible with the action of G_F , but contrary to the unramified case, that inclusion is not simplicial. (Note that there exist isomorphisms of simplicial complexes between X_E and X_F , but these isomorphisms are neither canonical nor useful for our purpose.) For that reason, we work most of the time with the geometric realization \mathcal{B}_F (resp. \mathcal{B}_E) of X_F (resp. X_E).

We have an inclusion $\mathcal{B}_F \subset \mathcal{B}_E$, and for every $x \in X_F$, x has the same geometric realization in both \mathcal{B}_F and \mathcal{B}_E . Once again, the inclusion is not simplicial: a facet of \mathcal{B}_F is usually the (disjoint) union of several facets of \mathcal{B}_E of various types. Moreover, when E/F is tamely ramified, \mathcal{B}_F is precisely the set of Γ -stable points of \mathcal{B}_E ; this is not true when E/F is wildly ramified.

For every facet D of X_E (resp. X_F), we denote by $R(D)$ its geometric realization in \mathcal{B}_E (resp. \mathcal{B}_F). Similarly, if A is an apartment of X_E (resp. X_F), we denote by $R(A)$ its geometric realization in \mathcal{B}_E (resp. \mathcal{B}_F). Note that D can be a facet of both X_E and X_F at the same time only if it is a vertex, and A cannot be an apartment of both X_E and X_F at the same time, hence there is no ambiguity with the notation.

Since G_E and G_F have the same semisimple rank, every apartment \mathcal{A} of \mathcal{B}_F is also an apartment of \mathcal{B}_E . Note that the apartments A_E of X_E and A_F of X_F whose geometric realization is \mathcal{A} are different; we though have the (nonsimplicial)

set equality $A_F = A_E \cap X_F$. We denote by \mathcal{A}_0 the apartment of \mathcal{B}_F (and also of \mathcal{B}_E) associated to T_0 , and by $A_{0,E}$ (resp. $A_{0,F}$) the apartment of X_E (resp. X_F) whose geometric realization is \mathcal{A}_0 .

For every subset S of \mathcal{B}_E , let $K_{S,E}$ (resp $K_{S,F}$) be the connected fixator of S in G_E (resp. G_F); this is an open compact subgroup of G_E (resp. G_F). If D is a facet of X_E (resp. X_F), we also write $K_{D,E}$ (resp. $K_{D,F}$) for $K_{R(D),E}$ (resp. $K_{R(D),F}$). If now X is any subset of X_E (resp. X_F), we define $K_{X,E}$ (resp. $K_{X,F}$) as the intersection of the $K_{x,E}$ (resp. $K_{x,F}$), $x \in X$; it is easy to check that this definition is consistent with the previous one when X is a facet. Finally, if T is a maximal torus of G defined over E (resp. F), we denote by $K_{T,E}$ (resp. $K_{T,F}$) the maximal compact subgroup of T_E (resp. T_F); it is easy to check that if A_E (resp. A_F) is the apartment of X_E (resp. X_F) associated to T , we have $K_{T,E} = K_{A_E,E}$ (resp. $K_{T,F} = K_{A_F,F}$).

We say that a vertex x of X_E (resp. X_F) is *E-special* (resp. *F-special*) if x is a special vertex of X_E (resp. X_F), or in other words, if the root system of the reductive quotient $K_{x,E}/K_{x,E}^0$ (resp. $K_{x,F}/K_{x,F}^0$) relative to some maximal torus, where $K_{x,E}^0$ (resp. $K_{x,F}^0$) is the pro-unipotent radical of $K_{x,E}$ (resp. $K_{x,F}$), is the full root system Φ of G_E (resp. G_F). Special vertices always exist (see [3, §3, cor. to proposition 11] for example). We also say that a vertex of \mathcal{B}_E (resp. \mathcal{B}_F) is *E-special* (resp. *F-special*) if it is the geometric realization of some *E-special* (resp. *F-special*) vertex of E (resp. F).

It is easy to prove that every *F-special* vertex of X_F is also *E-special*, but the converse is not true: *E-special* vertices of X_F are not necessarily *F-special*, and some *E-special* vertices of X_E do not even belong to X_F .

We fix once for all a *F-special* vertex x_0 of $A_{0,E}$. We can identify \mathcal{A}_0 with the \mathbb{R} -affine space $(X_*(T)/X_*(Z)) \otimes \mathbb{R}$, where Z is the center of G , by setting the origin at x_0 ; the elements of Φ are then identified, via the standard duality product $\langle \cdot, \cdot \rangle$ between $X^*(T)$ and $X_*(T)$, with affine forms on \mathcal{A}_0 , and the walls of \mathcal{A}_0 as an apartment of \mathcal{B}_F (resp. \mathcal{B}_E) are the hyperplanes satisfying an equation of the form $\alpha(x) = c$, with $\alpha \in \Phi$ and $c \in \mathbb{Z}$ (resp. $c \in \frac{1}{2}\mathbb{Z}$). Moreover, every facet D of $A_{0,F}$ (resp. $A_{0,E}$) is determined by a function f_D from Φ to \mathbb{Z} (resp. $\frac{1}{2}\mathbb{Z}$) the following way: for every $\alpha \in \Phi$, $f_D(\alpha)$ is the smallest element of \mathbb{Z} (resp. $\frac{1}{2}\mathbb{Z}$) which is greater or equal to $\alpha(x)$ for every $x \in R(D)$. If D is a facet of $A_{0,F}$ (resp. $A_{0,E}$), f_D satisfies the following properties:

- f_D is a concave function, or in other words:
 - for every $\alpha \in \Phi$, $f(\alpha) + f(-\alpha) \geq 0$;
 - for every $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$, $f(\alpha + \beta) \leq f(\alpha) + f(\beta)$.
- for every $\alpha \in \Phi$, $f(\alpha) + f(-\alpha) \leq 1$ (resp. $\frac{1}{2}$);
- if D is a *F-special* (resp. *E-special*) vertex, then for every $\alpha \in \Phi$, $f(\alpha) + f(-\alpha) = 0$. If D is a chamber of X_F (resp. X_E), then for every $\alpha \in \Phi$, $f(\alpha) + f(-\alpha) = 1$ (resp. $\frac{1}{2}$).

Note that if D is a E -special vertex of X_E belonging to X_F but not F -special, the functions f_D attached to D as a facet of respectively X_E and X_F are different. For these particular vertices, we have to denote by respectively $f_{D,E}$ and $f_{D,F}$ these two functions. In all other cases, either D is a facet of only one of the two buildings or the concave functions are identical, and there is then no ambiguity with the notation f_D .

We denote by $C_{0,F}$ the chamber of X_F such that $K_{C_{0,F}}$ is the standard Iwahori subgroup of G_F (relative to T_0 , Φ^+ and x_0), or in other words the chamber of \mathcal{A}^0 whose associated concave function $f_{C_{0,F}}$ is defined by $f(\alpha) = 0$ (resp. $f(\alpha) = 1$) for every positive (resp. negative) α . We also set $\mathcal{C}_{0,F} = R(C_{0,F})$.

For every $\alpha \in \Phi$, let U_α be the root subgroup of G attached to α , and let ϕ_α be the valuation on $U_{\alpha,E}$ defined the following way: for every $u \in U_{\alpha,E}$, $\phi_\alpha(u)$ is the largest element of $\frac{1}{2}\mathbb{Z}$ such that u fixes the half-plane $\alpha(x) \leq \phi_\alpha$ of \mathcal{A}_0 pointwise. (By convention, we have $\phi_\alpha(1) = +\infty$.) Obviously, the valuation on $U_{\alpha,F}$ defined in a similar way is just the restriction of ϕ_α to $U_{\alpha,F}$, hence there is no ambiguity in the notation. The quadruplet $(G, T_0, (U_\alpha)_{\alpha \in \Phi}, (\phi_\alpha)_{\alpha \in \Phi})$ is a valued root datum in the sense of Bruhat-Tits (see [7, I. 6.2]).

Now we give the definition of the harmonic cochains that we will be using throughout the whole paper. Let Ch_E be the set of chambers of X_E , and let $\mathcal{H}(X_E)$ be the vector space of harmonic cochains on Ch_E , or in other words the space of applications from Ch_E to \mathbb{C} satisfying the following condition (called the *harmonicity condition*): for every facet D of codimension 1 of X_E , we have:

$$\sum_{C \in Ch_E, D \subset C} f(C) = 0.$$

The group G_E acts naturally on $\mathcal{H}(X_E)$ by $g.f : C \mapsto f(g^{-1}C)$. For every subgroup L of G_E , we denote by $\mathcal{H}(X_E)^L$ the subspace of L -invariant elements of $\mathcal{H}(X_E)$. We also denote by $\mathcal{H}(X_E)^\infty$ the subspace of smooth elements of $\mathcal{H}(X_E)$, which is the union of the $\mathcal{H}(X_E)^K$, with K running over the set of open compact subgroups of G .

3. THE CHARACTERS χ AND ε

Let χ be the character of G_F defined the following way: let ρ be the half-sum of the elements of Φ^+ . By [4, §I, proposition 29], for every element $\alpha^\vee \in \Phi^\vee$, $\langle \rho, \alpha^\vee \rangle$ is an integer, hence $\langle 2\rho, \alpha^\vee \rangle$ is even; we deduce from this that for every quadratic character η of F^* , the character $\eta \circ 2\rho$ of $(T_0)_F$ is trivial on the subgroup of $(T_0)_F$ generated by the images of the α^\vee , which is the group $(T_0)_F \cap G_{F,der}$, where $G_{F,der}$ is the derived group of G_F ; $\eta \circ 2\rho$ then extends in a unique way to a quadratic character of $(T_0)_F G_{F,der} = G_F$; it is easy to check that such a character does not depend on the choice of T_0 , B_0 and Φ^+ .

Let $\varepsilon_{E/F}$ be the quadratic character of F^* associated to the extension E/F : for every $x \in F^*$, $\varepsilon_{E/F}(x) = 1$ if and only if x is the norm of an element of E^* . Let χ be the character $\varepsilon_{E/F} \circ 2\rho$ extended to G_F .

Proposition 3.1. *The character χ of G_F is the Prasad character of G_F relative to the extension E/F .*

According to [10, section 2], the Prasad character is of the form $\varepsilon_{E/F} \circ \chi_0$ for some $\chi_0 \in X^*(G)$, and we deduce from [10, lemma 3.1] that χ_0 is trivial if and only if $\rho \in X^*(T)$. On the other hand, since $\varepsilon_{E/F}$ is of finite order, $\varepsilon_{E/F} \circ \chi_0$ factors through a subgroup of finite index G_0 of G_F , and in particular the proposition holds when the quotient G_F/G_0 is cyclic. By [4, plates I to IX, (VIII)], that condition is satisfied as soon as Φ is not of type D_d with d even,

Assume then Φ is of type D_d , with $d = 2n$ being even. By [4, plate IV, (VII)], we have:

$$\rho = \left(\sum_{i=1}^{2n-2} \left(2ni - \frac{i(i-1)}{2} \alpha_i \right) \right) + \frac{n(2n-1)}{2} (\alpha_{2n-1} + \alpha_{2n}).$$

When n is even, ρ belongs to $X^*(T)$ and χ is then trivial, hence the proposition holds again. Assume now n is odd. Then by [10, section 5] again, we have for every $g \in G_F$:

$$\chi(g) = \varepsilon_{E/F} \circ (\alpha_{2n-1} + \alpha_{2n})(g),$$

and using the above expression of ρ , we obtain, given that $\varepsilon_{E/F}$ is quadratic:

$$\begin{aligned} \varepsilon_{E/F} \circ 2\rho(g) &= \varepsilon_{E/F} \circ \sum_{i=1}^{2n-2} (4ni - i(i-1)) \alpha_i(g) + \varepsilon_{E/F} \circ n(2n-1) (\alpha_{2n-1} + \alpha_{2n})(g) \\ &= \varepsilon_{E/F} \circ (\alpha_{2n-1} + \alpha_{2n})(g). \end{aligned}$$

Hence χ and $\varepsilon_{E/F} \circ \chi_0$ are equal, as desired. \square

Note that, since we are dealing with a ramified extension here, the subgroup G_0 of G_F we are using in the above proof is not the same as in [10], but this is of no importance: once we are reduced to a finite group, that group, up to a canonical isomorphism, depends only on Φ and not on E and F , and the proof works exactly the same way in the ramified and unramified cases.

Let now ε be the character of G_E defined the following way: let g be an element of G_E and let C be a chamber of X_E . Since X_E is labellable (see for example [6, IV, proposition 1]), there exists a canonical bijection λ between the vertices of C and the vertices of gC , and the application $x \mapsto g\lambda^{-1}(x)$ is then a permutation of the set of vertices of gC . We set $\varepsilon(g)$ to be the signature of that permutation; it is easy to check (see [5, lemma 2.1 (i) and (ii)]) that ε is actually a character of G_E and that it does not depend on the choice of C .

Let $(\pi_E, \mathcal{H}(X_E)^\infty)$ be the representation of G_E defined the following way: for every $g \in G_E$ and every $f \in \mathcal{H}(X_E)^\infty$, we have:

$$\pi_E(g)f : C \in Ch_E \longmapsto \varepsilon(g)f(g^{-1}C).$$

By [5, proposition 3.2], the representation $(\pi_E, \mathcal{H}(X_E)^\infty)$ of G_E is equivalent to $St_E \otimes \varepsilon$. On the other hand, when E/F is ramified, we have:

Proposition 3.2. *The character ε is trivial on G_F .*

Let $K_{T_0,F}$ be the maximal compact subgroup of $(T_0)_F$, and let $X_{T_0,F}$ be the subgroup of T_0 whose elements are the $\xi(\varpi_F)$, with $\xi \in X_*(T_0)$. From the decomposition $F^* = \varpi^\mathbb{Z} \mathcal{O}_F^*$ of F^* , we deduce the following decomposition of $(T_0)_F$:

$$(T_0)_F = K_{T_0,F} X_{T_0,F}.$$

Since $G_F = G_{F,der}(T_0)_F$, we finally obtain the following decomposition:

$$G_F = G_{F,der} K_{T_0,F} X_{T_0,F}.$$

Now consider the restriction of the character ε to G_F . Since $G_{F,der}$ is contained in $G_{E,der}$, ε is trivial on $G_{F,der}$; since $K_{T_0,E}$ fixes every chamber of $(A_0)_E$ pointwise, ε is also trivial on that group, and in particular on $K_{T_0,F}$; finally, $X_{T_0,F}$ is generated by the $\xi(\varpi_F)$, $\xi \in X_*(T_0)$; since ϖ_F is the product of ϖ_E^2 with some element x of \mathcal{O}_E^* , for every $\xi \in X_*(T_0)$, we have $\xi(\varpi_F) = \xi(\varpi_E)^2 \xi(x)$, and since $\xi(x) \in K_{T_0,E}$ and ε is quadratic and trivial on $K_{T_0,E}$, we obtain $\varepsilon(\xi(\varpi_F)) = 1$. Therefore, ε is trivial on $X_{T_0,F}$, hence on G_F and the proposition is proved. \square

Corollary 3.3. *The restriction to G_F of the representation π'_E given by the natural action of G_E on $\mathcal{H}(X_E)$ is isomorphic to the restriction of St_E .*

Corollary 3.4. *For every character χ of G_F , $\text{Hom}_{G_F}(St_E, \chi)$ and $\text{Hom}_{G_F}(\pi'_E, \chi)$ are canonically isomorphic.*

This last corollary proves that when E/F is ramified, the χ -distinctions of St_E and π'_E with respect to G_E/G_F are two equivalent problems. For that reason, in the sequel, we work with π'_E instead of St_E .

4. THE ANISOTROPY CLASS OF A CHAMBER

In this section, we classify the chambers of X_E according to the F -anisotropy classes of E -split F -tori of G , at least when E/F is tamely ramified.

First we have to prove that for every chamber C , there exists a E -split maximal F -torus of G such that C is contained in the apartment of X_E associated to T ; this is an immediate consequence of the following result, which is the tamely ramified equivalent of [5, Lemma A.2]:

Proposition 4.1. *Assume E/F is tamely ramified. Let C be any chamber of X_E ; there exists a Γ -stable apartment of X_E containing both C and $\gamma(C)$.*

This is simply a particular case of [12, proposition 3.8]. \square

Note that the result of [12] is also valid when E/F is unramified, but only when the residual characteristic of F is odd; this is the reason why we used a different proof for [5, lemma A.2], which works for any F .

Note also that the above proposition is not true when E/F is wildly ramified. As a counterexample, consider a Γ -stable chamber C of X_E whose geometric realization is not contained in \mathcal{B}_F ; such chambers actually exist when E/F is wildly ramified. Let A be a Γ -stable apartment of X_E containing C ; since Γ fixes a chamber of A , it fixes A pointwise, which implies that A is associated to some F -split torus of G , and we must then have $R(A) \subset \mathcal{B}_F$; since $R(C) \subset R(A)$ is already not contained in \mathcal{B}_F by hypothesis, we reach a contradiction.

We now classify E -split F -tori of G according to the roots of G intervening in their anisotropic component. Recall that two elements α and β of Φ are said to be *strongly orthogonal* if they are orthogonal (or in other words, if $\langle \alpha, \beta^\vee \rangle = 0$) and $\alpha + \beta$ is not an element of Φ . First we prove some lemmas.

Lemma 4.2. *Assume α and β are strongly orthogonal. Then $-\alpha$ and β are also strongly orthogonal.*

If α and β are orthogonal, then $-\alpha$ and β are orthogonal as well. Moreover, let $s_\alpha \in W$ be the reflection associated to α ; we have $s_\alpha(\alpha + \beta) = -\alpha + \beta$, and since $\alpha + \beta \notin \Phi$, $-\alpha + \beta$ cannot belong to Φ either and the lemma is proved. \square

Lemma 4.3. *Let α, β be two elements of Φ . If α and β are orthogonal and at least one of them is long, then they are strongly orthogonal.*

(By convention, if Φ is simply-laced, all of its elements are considered long.)

It is easy to check (it is nothing else than the good old Pythagorean theorem) that when α and β are orthogonal, $\alpha + \beta$ is strictly longer than either of them. Hence since Φ is reduced, $\alpha + \beta$ can be a root only if α and β are both short. The lemma follows. \square

Lemma 4.4. *The following assertions are equivalent:*

- *there exists $w \in W$ such that $w(\alpha) = -\alpha$ for every $\alpha \in \Phi$;*
- *there exists a subset Σ of Φ whose cardinality is the rank d of Φ and such that two distinct elements of Σ are always strongly orthogonal.*

Moreover, when Σ exists, it is unique up to conjugation by an element of W .

Assume $w \in W$ is such that $w(\alpha) = -\alpha$ for every $\alpha \in \Phi$. We prove the first implication by induction on the rank d of Φ ; we prove in addition that, if Σ satisfies the conditions of the second assertion, we have:

$$w = \prod_{\alpha \in \Sigma} s_\alpha,$$

where for every α , s_α is the reflection associated to α . Note that since the elements of Σ are all orthogonal to each other, the s_α commute, hence the above product can be taken in any order.

The case $d = 0$ is trivial: assume $d > 0$. Let α_0 be the highest root in Φ^+ ; by [4, proposition 25 (iii)], α_0 is always a long root. Consider the elementary reflection $s_{\alpha_0} \in W$ associated to α_0 ; the set Φ_{α_0} of roots β of Φ such that $s_{\alpha_0}w(\beta) = -\beta$ is precisely the set of elements of Φ which are orthogonal to α_0 , hence strongly orthogonal to α_0 by lemma 4.3. Moreover, Φ_{α_0} is a closed and symmetrical subset of Φ , hence a root subsystem of Φ , of rank strictly smaller than d , and for every $\beta \in \Phi^+$, we have $s_{\alpha_0}w(\beta) = -\beta + \langle \beta, \alpha_0^\vee \rangle \alpha_0$, which is negative if and only if $\beta \in \Phi_{\alpha_0}$; we can thus apply the induction hypothesis to Φ_{α_0} and $s_{\alpha_0}w$ to obtain a subset Σ' of Φ_{α_0} satisfying the conditions of the second assertion (relatively to Φ_{α_0} and such that we have:

$$s_{\alpha_0}w = \prod_{\beta \in \Sigma'} s_\beta.$$

Finally, we set $\Sigma = \Sigma' \cup \{\alpha_0\}$.

Note that Φ_{α_0} may be reducible; in such a case, we apply the induction hypothesis to each one of its irreducible components and take as Σ' the union of the sets of roots we obtain that way, given that two elements of Φ_{α_0} which belong to different irreducible components are always strongly orthogonal.

It only remains to check that Σ contains d elements. Since these elements must be linearly independent, Σ cannot contain more than d of them. Assume it contains less than d elements; there exists then $\beta \in \Phi$ which is not a linear combination of elements of Σ . On the other hand, it is easy to check (for example by decomposing it into a sum of terms of the form $s_\alpha(\beta') - \beta'$ (which is a multiple of α), with $\alpha \in \Sigma$ and $\beta' \in \Phi$) that $w(\beta) - \beta$ is a linear combination of elements of Σ ; we then cannot have $w(\beta) = -\beta$, hence a contradiction.

Conversely, let Σ be a subset of Φ satisfying the conditions of the second assertion; set:

$$w = \prod_{\alpha \in \Sigma} s_\alpha.$$

Since the elements of Σ are all orthogonal to each other, we must have $w(\alpha) = -\alpha$ for every $\alpha \in \Sigma_T$. Moreover, since the cardinality of Σ_T is d and its elements are linearly independent, they generate $X^*(T) \otimes \mathbb{Q}$ as a \mathbb{Q} -vector space, and every element of Φ is then a linear combination of them, which implies, since w extends to a linear automorphism of $X^*(T) \otimes \mathbb{Q}$, that we have $w(\alpha) = -\alpha$ for every $\alpha \in \Phi$, as required. \square

By [4, plates I to IX, (XI)], the conditions of the above proposition are satisfied for every Φ except the following ones:

- Φ if type A_d with $d > 1$;
- Φ of type D_d with d odd;
- Φ of type E_6 .

Now we separate the E -split maximal tori of G into F -anisotropy classes. The reductive subgroups L_0 and L of G that we introduce in proposition 4.5 and its proof will be of some use later (see section 6).

Assume E/F is tamely ramified. Let \mathcal{A} be a Γ -stable apartment of \mathcal{B}_E . Since E/F is tamely ramified, \mathcal{A}^Γ is contained in \mathcal{B}_F ; by [7, I. proposition 2.8.1], there exists an apartment \mathcal{A}' of \mathcal{B}_F containing \mathcal{A}^Γ , and by eventually conjugating \mathcal{A} by a suitable element of G_F , we can assume $\mathcal{A}' = \mathcal{A}_0$. Let T be the E -split maximal torus of G_E associated to \mathcal{A} ; the F -split component T_s of T is then contained in T_0 .

Proposition 4.5. *Let a be the dimension of the F -anisotropic component of T . With the above hypotheses, there exists a unique (up to conjugation) subset Σ_T of Φ , of cardinality a , such that:*

- *T is G_F -conjugated to some maximal torus of G contained in the reductive subgroup L_0 of G generated by T_0 and the root subgroups $U_{\pm\alpha}$, $\alpha \in \Sigma_T$, and F -elliptic in L_0 ;*
- *if $\alpha, \beta \in \Sigma_T$, then α and β are strongly orthogonal.*

Conversely, for every $\Sigma \subset \Phi$ satisfying the second condition, there exists an E -split maximal torus T of G defined over F such that we can choose $\Sigma_T = \Sigma$.

Let \mathcal{A}^Γ be the affine subspace of Γ -fixed points of \mathcal{A} ; since T_0 contains the split component of T , every facet of maximal dimension of \mathcal{A}^Γ is contained in the closure of some chamber of \mathcal{A}^0 . Let \mathcal{D} be such a facet; by eventually conjugating T by a suitable element of G_F , we can assume that \mathcal{D} is contained in the closure of $R(C_{0,F})$.

Moreover, T is contained in the centralizer $Z_G(T_s)$ of T_s in G , hence if Σ_T exists, we can assume that the root subgroups $U_{\pm\alpha}$, $\alpha \in \Sigma_T$, are also all contained in $Z_G(T_s)$. Hence by replacing G by $Z_G(T_s)/T_s$, we can assume that T is F -anisotropic, which implies that \mathcal{D} is a vertex of \mathcal{B}_E contained in \mathcal{B}_F . (Note that \mathcal{D} is not necessarily a vertex of \mathcal{B}_F .) The existence of a subset Σ_T of Φ of cardinality d satisfying the strong orthogonality condition is then a consequence of lemma 4.4, but we still have to prove that such a Σ_T satisfies the first condition as well.

Since T is E -split, there exists $g \in G_E$ such that $gTg^{-1} = T_0$; the conjugation by g^{-1} sends then Φ to the root system of G relative to T . Since \mathcal{A}^Γ consists of a single point, the action of the nontrivial element γ of Γ on \mathcal{A} is the central symmetry relative to that point. This means in particular that for every $\alpha \in \Phi$, $\gamma(Ad(g^{-1})\alpha) = -Ad(g^{-1})\alpha$.

Let L_0 be the subgroup of G generated by T_0 and the root subgroups $U_{\pm\alpha}$, $\alpha \in \Sigma_T$. Set $L = gL_0g^{-1}$; L is then the subgroup of G generated by T and the root subgroups $gU_{\pm\alpha}g^{-1}$, $\alpha \in \Sigma_T$. This group is a closed E -split reductive subgroup of G of type $(A_1)^d$; moreover, for every $\alpha \in \Sigma_T$, since $\gamma(Ad(g^{-1})\alpha) = -Ad(g^{-1})\alpha$, we have $\gamma(gU_\alpha g^{-1}) = gU_{-\alpha}g^{-1}$; we deduce from this that L is Γ -stable, hence defined over F . To prove the first assertion of the proposition, we only have to prove that L and L_0 are G_F -conjugates.

We first prove the following lemma:

Lemma 4.6. *The group L is F -split.*

Since the elements of Σ_T are all strongly orthogonal to each other, L is F -isogeneous to the direct product of d semisimple and simply-connected groups of type A_1 , namely the groups generated by the $U_{\pm \text{Ad}(g)\alpha}$ for every $\alpha \in \Sigma$; moreover, since for every $\alpha \in \Sigma$, γ swaps $\text{Ad}(g)\alpha$ and $-\text{Ad}(g)\alpha$, every such component is Γ -stable. On the other hand, by [20, 17.1], there are exactly two simply-connected groups of type A_1 defined over F : the split group SL_2 , and its unique nonsplit F -form, whose group of F -points is isomorphic to the group of the norm 1 elements of the unique quaternionic division algebra over F (these groups are the only inner F -forms of SL_2 by [20, proposition 17.1.3], and by the remark made at the beginning of [20, 17.1.4], SL_n can have outer forms only if $n \geq 3$). Let F' be the unique quadratic unramified extension of F ; these groups are both F' -split, which proves that L must be F' -split as well.

Let T' be a maximal F' -split F -anisotropic torus of G contained in L and let $K_{T',F}$ be the maximal compact subgroup of T'_F . By [11, theorem 3.4.1], there exists a pair (K, \mathbb{T}') , with K being a maximal parahoric subgroup of G_F and \mathbb{T}' being a maximal k_F -torus in the quotient $\mathbb{G} = K/K^0$, k_F -anisotropic modulo the center of \mathbb{G} , such that $K_{T',F} \subset K$ and \mathbb{T}' is the image of $K_{T',F}$ in \mathbb{G} ; moreover, the dimension of the k_F -anisotropic component of \mathbb{T}' is the same as the dimension of the F -anisotropic component of T' , which implies that \mathbb{G} is of semisimple rank d and \mathbb{T}' is k_F -anisotropic.

Consider now the image \mathbb{L} of $L_F \cap K$ in \mathbb{G} ; \mathbb{L} is the group of k_F -points of a reductive k_F -group k_F -isogeneous to the direct product of d k_F -split simply-connected groups of type A_1 . Since by [8, 1.17], every group over a finite field is quasisplit, and since the only quasisplit simply-connected group of type A_1 over any field is SL_2 , which is split, \mathbb{L} is isogeneous to a k_F -split group, hence is k_F -split itself and contains a k_F -split maximal torus \mathbb{T}'' . Let I be an Iwahori subgroup of G_F contained in K whose image in \mathbb{G} contains \mathbb{T}'' ; considering the Iwahori decomposition of I (or alternatively, using [11, theorem 3.4.1] again), we see that there exists a maximal torus T'' of G whose maximal compact subgroup $K_{T''}$ is contained in I and such that \mathbb{T}'' is the image of $K_{T''}$ in \mathbb{G} , and T'' must then be F -split. Hence L is F -split, as desired. \square

Now we go back to the proof of proposition 4.5. We prove L is G_F -conjugated to L_0 , and also the unicity of Σ_T up to conjugation. By eventually conjugating L by some element of G_F , we can assume that it contains T_0 ; L is then generated by T_0 and the $U_{\pm\alpha}$, with α belonging to some set Σ' satisfying the strong orthogonality condition, and L and L_0 are G_E -conjugated by some element n of the normalizer of T_0 in G_E , which implies that Σ_T and Σ' are W -conjugates. Moreover, since G is F -split, it is possible to choose n as an element of G_F , hence L and L_0 are G_F -conjugates as well and the first assertion of proposition 4.5 is proved.

Now we prove the second assertion. Let Σ be any subset of Φ^+ such that every $\alpha \neq \beta \in \Sigma$ are strongly orthogonal. The reductive subgroup L of G generated

by T_0 and the $U_{\pm\alpha}$, $\alpha \in \Sigma$, is then of type A_1^a , where a is the cardinality of Σ ; by quotienting L by its center and considering separately every one of its irreducible components, we are reduced to the case where L is a simple group of type A_1 , hence isogeneous to SL_2 ; according to a well-known result about SL_2 , since E/F is quadratic and separable, L contains a 1-dimensional E -split F -anisotropic torus, as required (for example, when E/F is tamely ramified, the group of elements of SL_2 of the form $\begin{pmatrix} a & b \\ -\varpi b & a \end{pmatrix}$, where ϖ is a uniformizer of F which is the square of some uniformizer of E). \square

More generally, since every E -split F -torus T of G is G_F -conjugated to some torus T' whose F -split component is contained in T_0 , by the previous proposition, we can attach to T a subset Σ_T of Φ^+ , defined up to conjugation, which is the subset attached to T' by that proposition. The class of Σ_T is called the *F -anisotropy class* of T .

Note that, although the F -anisotropy classes are parametrized by the conjugacy classes of subsets of strongly orthogonal elements of Φ , in the sequel, by a slight abuse of notation, we will often designate an F -anisotropy class by one of the representatives of the corresponding conjugacy class; more precisely, we will say "the F -anisotropy class Σ " instead of "the F -anisotropy class corresponding to the conjugacy class of subsets of strongly orthogonal elements of Φ which contain Σ ".

Note also that, as we will see later, two E -split F -tori belonging to the same F -anisotropy class are not necessarily G_F -conjugates; we though have the following result:

Proposition 4.7. *Assume E/F is tamely ramified. Let T, T' be two E -split maximal F -tori of G belonging to the same F -anisotropy class Σ and let \mathcal{A} (resp. \mathcal{A}') be the Γ -stable apartment of \mathcal{B}_E associated to T (resp. T'). Then the affine subspaces \mathcal{A}^Γ and \mathcal{A}'^Γ are $G_{F,der}$ -conjugates.*

Since E/F is tamely ramified, \mathcal{A}^Γ and \mathcal{A}'^Γ are contained in \mathcal{B}_F , and by eventually conjugating T and T' by elements of $G_{F,der}$, we can assume that they are both contained in \mathcal{A}_0 ; they are then conjugated by some element n of the normalizer of T_0 in $G_{E,der}$. Moreover, since T_0 is F -split, every element of the Weyl group of G/T_0 admits representatives in G_F , and even in $G_{F,der}$ since the Weyl groups of G_F and $G_{F,der}$ are the same; hence by eventually conjugating T again, we may assume $n \in T_0 \cap G_{E,der}$. Finally, we have $T_0 \cap G_{E,der} = (K_{T_0,E} \cap G_{E,der})(X_{T_0,E} \cap G_{E,der})$, where $X_{T_0,E}$ is the subgroup of T_0 generated by the $\xi(\varpi_E)$, $\xi \in X_*(T_0)$, and $K_{T_0,E}$ fixes \mathcal{A}^Γ pointwise; we thus may assume that n belongs to $X_{T_0,E} \cap G_{E,der}$, which is, since G_{der} is simply-connected, the subgroup of $X_{T_0,E}$ generated by the $\alpha^\vee(\varpi_E)$, $\alpha^\vee \in \Phi^\vee$. In such a case, the split components of T and T' are both contained in T_0 and conjugated by an element

of T_0 , hence identical; we thus can assume that Σ is contained in the root subsystem of the elements of Φ whose restriction to that common split component is trivial.

We now prove the result with n being of the form $\alpha^\vee(\varpi_E)$ for some α^\vee ; the general case follows by an easy induction. If $\langle \beta, \alpha^\vee \rangle = 0$ for every $\beta \in \Sigma$, then $\mathcal{A}^\Gamma = \mathcal{A}'^\Gamma$ and there is nothing to prove. If $\langle \beta, \alpha^\vee \rangle$ is odd for some $\beta \in \Sigma$, then either \mathcal{A}^Γ or \mathcal{A}'^Γ , say for example \mathcal{A}^Γ , is contained in some hyperplane of \mathcal{A}_0 which is a wall in \mathcal{B}_F and whose associated roots are $\pm\beta$; on the other hand, if L is the reductive subgroup of G associated to T as in proposition 4.5 and if L_β is the subgroup of L generated by the root subgroups $U_{\pm\beta}$, $T \cap L_\beta$ is then split, hence T is of anisotropy class strictly contained in Σ , which leads to a contradiction. Hence $\langle \beta, \alpha^\vee \rangle$ must be even for every $\beta \in \Sigma$.

Assume now $\langle \beta, \alpha^\vee \rangle$ is even for every $\beta \in \Sigma$ and nonzero for at least one β ; that nonzero $\langle \beta, \alpha^\vee \rangle$ must then be equal to ± 2 . As a consequence, there exists a wall \mathcal{H} of the apartment \mathcal{A}_0 of \mathcal{B}_F which separates \mathcal{A}^Γ from \mathcal{A}'^Γ and contains neither of them; if we assume the converse, we reach the same contradiction as above. Let $s_{\mathcal{H}}$ be the orthogonal symmetry with respect to \mathcal{H} ; we obviously have $s_{\mathcal{H}}(\mathcal{A}) = \mathcal{A}'$. On the other hand, \mathcal{H} being a wall in the building \mathcal{B}_F , the element of the affine Weyl group of T_0 corresponding to $s_{\mathcal{H}}$ admits representatives in $G_{F,der}$; the result follows. \square

Note that the above proof does not work in the wildly ramified case because \mathcal{A}^Γ and \mathcal{A}'^Γ are then not contained in \mathcal{B}_F in general. The author conjectures that proposition 4.7 still holds in that case, though.

Now we want to divide Ch_E into F -anisotropy classes as well. Of course the Γ -stable apartment containing C , hence also the E -split maximal F -torus associated to it, are not unique, but we can still prove the following result:

Proposition 4.8. *Assume E/F is tamely ramified. Let C be any chamber of X_E and let A and A' be two Γ -stable apartments of X_E containing C . Then the pairs (C, A) and (C, A') are $G_{F,der}$ -conjugates.*

When $R(C)$ is contained in \mathcal{B}_F , $R(A)$ and $R(A')$ must also be contained in \mathcal{B}_F , and [7, I. proposition 2.3.8] implies that they are then always $G_{F,der}$ -conjugates. Assume now $R(C)$ is not contained in \mathcal{B}_F and let g be an element of $G_{E,der}$ such that $gC = C$, $g\gamma(C) = \gamma(C)$ and $gA = A'$; such an element exists by [7, I. proposition 2.3.8] again. Moreover, we also have $\gamma(g)C = C$, hence $g \in K_{C \cap \gamma(C), E}$, and $\gamma(g)A = A'$; if we set $h = \gamma^{-1}(g)g$, we then have $hC = C$ and $hA = A$, which implies, if T is the E -split maximal torus of G attached to A , that $h \in K_{T,E}$.

Since $C \cup \gamma(C)$ contains a chamber of A , $K_{C \cup \gamma(C), E}$ is contained in an Iwahori subgroup of G_E , and since it contains $K_{T,E}$, we have $K_{C \cup \gamma(C), E} = K^0 K_{T,E}$, where K^0 is the pro-unipotent radical of $K_{C \cup \gamma(C), E}$. By multiplying g by a suitable element of $K_{T,E}$ on the right, we may assume $g \in K^0$, which implies $h \in K^0 \cap K_{T,E}$. On the other hand, by [9, corollary 1], the cohomology group $H^1(\Gamma, K^0 \cap K_{T,E})$ is trivial, which implies that since $h = \gamma^{-1}(g)g$ satisfies $h\gamma(h) = 1$, and

thus defines a 1-cocycle of $\Gamma = \{1, \gamma\}$, it also defines a 1-coboundary of that same group, hence must admit a decomposition of the form $h = \gamma(h')h'^{-1}$, with h' being an element of $K_{T,E} \cap K^0$; hence $gh' = \gamma(gh')$, which implies that gh' is an element of $G_{F,der}$ such that $gh'\mathcal{C} = \mathcal{C}$ and $gh'\mathcal{A} = \mathcal{A}'$, as desired. \square

Note that the tame ramification hypothesis is needed for the above proof because it is used by [9, corollary 1], but the author believes that in the wildly ramified case, a similar result should hold for chambers of X_E contained in at least one Γ -stable apartment.

Corollary 4.9. *Assume E/F is tamely ramified. Let C and A be defined as in proposition 4.8, let T be the maximal E -split F -torus of G associated to A and let Σ_T be a subset of Φ attached to T as in proposition 4.5. Then up to conjugation, Σ_T does not depend on the choice of A .*

This is an obvious consequence of proposition 4.8. \square

In other words, the F -anisotropy class of the torus T associated to a Γ -stable apartment A of X_E containing C does not depend on the choice of A . We can now state the following definition:

Definition 4.10. *Let C be a chamber of X_E . The F -anisotropy class of C is the F -anisotropy class of the E -split maximal torus of G associated to any Γ -stable apartment of X_E containing C .*

5. THE SUPPORT OF THE $G_{F,der}$ -INVARIANT HARMONIC COCHAINS

In this section, we start the proof of theorem 1.2. In the unramified case (see [5, section 6]), in order to prove a similar result, we fix a chamber C_0 of $X_F \subset X_E$ and then, for every $C \in Ch_E$, we prove by induction on the combinatorial distance between C and X_F that for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$, $f(C)$ depends only on $f(C_0)$. In the ramified case, a similar approach would be to start from a chamber of X_E whose geometric realization is contained in \mathcal{B}_F ; it turns out that although that kind of approach works in the case of a group of type A_{2n} , in the other cases, f is identically zero on the set of such chambers and we have to find another starting point for our induction. For that reason, we start by determining the support of the elements of $\mathcal{H}(X_E)^{G_{F,der}}$. In particular, when Φ is not of type A_{2n} , we prove that their support coincides with some given anisotropy class of Ch_E , namely the one given by proposition 5.10.

5.1. The class Ch_\emptyset . First we consider the trivial F -anisotropy class Ch_\emptyset of Ch_E , or in other words the F -anisotropy class corresponding to $\Sigma = \emptyset$. When E/F is tamely ramified, a chamber C belongs to the trivial anisotropy class if and only if its geometric realization is contained in an apartment \mathcal{A} of \mathcal{B}_E whose associated torus is F -split, which is true if and only if $\mathcal{A} \subset \mathcal{B}_F$. When E/F is wildly ramified, we also define Ch_\emptyset as the set of chambers of X_E satisfying that property.

Contrary to the unramified case, the action of $G_{F,der}$ on Ch_\emptyset is not transitive, and we thus have to check that the space of the restrictions of elements of $\mathcal{H}(X_E)^{G_{F,der}}$ to Ch_\emptyset is of dimension at most 1. We start by the following lemma:

Lemma 5.1. *Let f be an element of $\mathcal{H}(X_E)^{G_{F,der}}$, and let C be a chamber of X_E such that $R(C)$ is contained in \mathcal{B}_F and that the geometric realization of at least one of its walls is contained in a codimension 1 facet of \mathcal{B}_F . Then $f(C) = 0$.*

Let C_F (resp. D_F) be a chamber (resp. a codimension 1 facet) of X_F such that $R(C_F)$ contains $R(C)$ (resp. $R(D_F)$ contains some wall $R(D)$ of $R(C)$), and let S be a set of representatives in $G_{F,der}$ of the quotient group $K_{D_F,F}/K_{C_F,F}$. Since C (resp. D) and C_F (resp. D_F) have the same closure in \mathcal{B}_F , we have $K_{C_F,F} = K_{C,F}$ (resp. $K_{D_F,F} = K_{D,F}$); moreover, since E/F is totally ramified, $K_{D_F,F}/K_{C_F,F} = K_{D,F}/K_{C,F}$ is isomorphic to $K_{D,E}/K_{C,E}$, hence the chambers gC , $g \in S$, are precisely the chambers of X_E containing D ; by the harmonicity condition, we then have $\sum_{g \in S} f(gC) = 0$. On the other hand, since f is $G_{F,der}$ -invariant, we have $f(gC) = f(C)$ for every $g \in S$, hence the result. \square

Now we determine which chambers of Ch_\emptyset do or do not satisfy the condition of the previous lemma.

Proposition 5.2. *The following conditions are equivalent:*

- *There exists a chamber C in Ch_\emptyset such that none of the walls of $R(C)$ is contained in a codimension 1 facet of \mathcal{B}_F . Moreover, every chamber of \mathcal{B}_F contains a unique chamber of \mathcal{B}_E satisfying that property;*
- *The root system Φ is of type A_{2n} , with n being a positive integer.*

Let C be any element of Ch_\emptyset , and set $\mathcal{C} = R(C)$. Assume C satisfies the condition of the proposition; since for every $g \in G_F$, gC satisfies it too, we can assume that \mathcal{C} is contained in $\mathcal{C}_{0,F}$. Let f_C be the concave function on Φ associated to C and let Δ'_C be the extended set of simple roots of Φ associated to C , which is the set of elements of Φ corresponding to the $d+1$ half-apartments of \mathcal{A}_0 whose intersection is \mathcal{C} . Since the walls of \mathcal{C} are not contained in any codimension 1 facet of \mathcal{B}_F , we must have $f_C(\alpha) \in \mathbb{Z} + \frac{1}{2}$ for every $\alpha \in \Delta'_C$. On the other hand, let $\Delta = \{\alpha'_1, \dots, \alpha'_d\}$ be a set of simple roots of Φ contained in Δ'_C and let $\alpha'_0 = -\sum_{i=1}^d \lambda_i \alpha_i$ be the remaining element of Δ'_C ; we have, with an obvious induction:

$$f_C(\alpha'_0) + \sum_{i=1}^d \lambda_i f_C(\alpha'_i) = f_C(\alpha'_0) + f_C\left(\sum_{i=1}^d \lambda_i \alpha'_i\right) = f_C(\alpha'_0) + f_C(-\alpha'_0) = \frac{1}{2}.$$

For every $i \in \{0, \dots, d\}$, we have $f_C(\alpha_i) \in \mathbb{Z} + \frac{1}{2}$, which implies:

$$\frac{1}{2} \in \mathbb{Z} + \left(1 + \sum_{i=1}^d \lambda_i\right) \frac{1}{2},$$

hence the integer $1 + \sum_{i=1}^d \lambda_i$ must be odd. By [4, §1, proposition 31], this integer is the Coxeter number of Φ , and by [4, plates I to IX, (III)], it is odd if and only if Φ is of type A_{2n} for some n ; the first implication of the proposition is then proved.

Now assume G is of type A_{2n} for some n . We prove that $\mathcal{C}_{0,F}$ contains exactly one chamber of \mathcal{B}_E satisfying the required condition; since that property translates by the action of G_F , every chamber of \mathcal{B}_F satisfies it as well.

Let Δ' be an extended set of simple roots of Φ and let \mathcal{C}' be the geometric realization of the chamber C' of A_0 defined by the concave function f such that:

- $f(\alpha) = \frac{1}{2}$ for every element α of Δ' different from some given one α_0 , and $f(\alpha_0) = \frac{1}{2} - n$;
- for every $\beta \in \Phi$, writing $\beta = \sum_{\alpha \in S} \alpha$ for a suitable proper subset S of Δ' (since Φ is of type A_d , such a subset exists, and it is unique), we have $f(\beta) = \sum_{\alpha \in S} f(\alpha)$.

Since $f(\alpha)$ is not an integer for any $\alpha \in \Delta'$, none of the walls of \mathcal{C}' are contained in codimension 1 facets of \mathcal{B}_F . The chamber \mathcal{C}' is generally not contained in $\mathcal{C}_{0,F}$, but is always conjugated by an element of G_F to some chamber \mathcal{C} contained in $\mathcal{C}_{0,F}$ which satisfies the same property.

Now we prove the unicity of \mathcal{C} . We use the notations of [4, plate I] (see also [4, §4.4]): Φ is a subset of a free abelian group X_0 of rank $2n + 1$ generated by elements $\varepsilon_1, \dots, \varepsilon_{2n+1}$ (this is the group denoted by L_0 in [4]; we rename it here to avoid confusion with the group L_0 of proposition 4.5), the elements of Φ are the ones of the form $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ with $i \neq j \in \{1, \dots, 2n + 1\}$, the elements of Φ^+ being the ones such that $i < j$, and W acts on X_0 by permutation of the ε_i . (The group X_0 is isomorphic to the character group of a maximal torus of GL_{2n+1} , and W is isomorphic to the symmetric group S_{2n+1} .)

Let $\mathcal{C} = R(C)$ be a chamber of \mathcal{B}_E contained in $\mathcal{C}_{0,F}$ and satisfying the required condition, and let f_C be the concave function associated to \mathcal{C} . Since \mathcal{C} is contained in $\mathcal{C}_{0,F}$, for every $\alpha \in \Phi^+$, we have $f_C(\alpha) \leq 0$ and $f_C(-\alpha) \leq 1$. On the other hand, we have $f_C(\alpha) + f_C(-\alpha) = \frac{1}{2}$, which implies $f_C(\alpha) \in \{-\frac{1}{2}, 0\}$ and $f_C(-\alpha) \in \{\frac{1}{2}, 1\}$.

Let Δ' be the extended set of simple roots associated to C ; since for every $\alpha \in \Delta'$, we have $f_C(\alpha) \in \mathbb{Z} + \frac{1}{2}$, we must have $f_C(\alpha) = -\frac{1}{2}$ if $\alpha > 0$ and $f_C(\alpha) = \frac{1}{2}$ if $\alpha < 0$. On the other hand, the sum of the $f_C(\alpha)$, $\alpha \in \Delta'$, is $\frac{1}{2}$; Δ' must then contain exactly n positive roots and $n + 1$ negative roots.

Now we examine more closely the elements of Δ' . Since W acts transitively on the set of all extended sets of simple roots of G , there exists an element w of W such that Δ' is the conjugate by w of the standard extended set of simple roots $\{\alpha_{12}, \alpha_{23}, \dots, \alpha_{2n+1,1}\}$, or in other words there exists a permutation σ of $\{1, \dots, 2n + 1\}$ such that $\Delta' = \{\alpha_{\sigma(1)\sigma(2)}, \alpha_{\sigma(2)\sigma(3)}, \dots, \alpha_{\sigma(2n+1)\sigma(1)}\}$.

Assume that for some i (with cycling indices), $\alpha_{\sigma(i)\sigma(i+1)}$ and $\alpha_{\sigma(i+1)\sigma(i+2)}$ are both positive. Then $f_C(\alpha_{\sigma(i)\sigma(i+2)}) = -\frac{1}{2} - \frac{1}{2} = -1$, which is impossible by the

previous remarks. Hence there must always be at least one negative root between two positive ones in the extended Dynkin diagram attached to Δ' , which is a cycle of length $2n + 1$. Since Δ' contains $n + 1$ negative roots and n positive roots, positive and negative roots must alternate on the diagram, except for two consecutive negative roots at some point. We can always choose σ in such a way that the consecutive negative roots are $\alpha_{\sigma(2n+1)\sigma(1)}$ and $\alpha_{\sigma(1)\sigma(2)}$; in that case, $\alpha_{\sigma(i)\sigma(i+1)}$ is positive if and only if i is even. We then easily obtain, for every $i < j$:

- if i and j are either both even or both odd, $f_C(\alpha_{\sigma(i)\sigma(j)}) = 0$, hence $\alpha_{\sigma(i)\sigma(j)}$ is positive, which implies $\sigma(i) < \sigma(j)$;
- if i is even and j is odd, $f_C(\alpha_{\sigma(i)\sigma(j)}) = -\frac{1}{2}$, hence $\alpha_{\sigma(i)\sigma(j)}$ is positive, which implies $\sigma(i) < \sigma(j)$;
- if i is odd and j is even, $f_C(\alpha_{\sigma(i)\sigma(j)}) = \frac{1}{2}$, hence $\alpha_{\sigma(i)\sigma(j)}$ is negative, which implies $\sigma(i) > \sigma(j)$.

In other words, the restriction of σ to the subset of even (resp. odd) elements of $\{1, \dots, 2n + 1\}$ is an increasing function, and for every i, j such that i is even and j odd, $\sigma(i) < \sigma(j)$. This is only possible if, for every i , $\sigma(2i) = i$ and $\sigma(2i + 1) = n + i + 1$, and Δ' is uniquely determined by these conditions. Since Δ' and the $f_C(\alpha)$, $\alpha \in \Delta'$, determine C , the unicity of C is proved. \square

Corollary 5.3. *When Φ is not of type A_{2n} for any n , for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$ and for every chamber C of Ch_\emptyset , $f(C) = 0$.*

When Φ is of type A_{2n} for some n , there exists a unique G_F -orbit Ch_c of chambers of X_E contained in Ch_\emptyset and such that the elements of $\mathcal{H}(X_E)^{G_{F,der}}$ are identically zero on $Ch_\emptyset - Ch_c$.

This is an immediate consequence of lemma 5.1 and proposition 5.2. In the case A_{2n} , the orbit Ch_c is the one described in the proof of proposition 5.2. \square

Let C_F be a chamber of X_F , and let C be the unique element of Ch_c whose geometric realization is contained in $R(C_F)$. We will call C the *central chamber* of C_F .

Corollary 5.4. *The space of the restrictions to Ch_\emptyset of the elements of $\mathcal{H}(X_E)^{G_{F,der}}$ is of dimension at most 1.*

This is an immediate consequence of the previous corollary. \square

5.2. The other anisotropy classes. Now we deal with the remaining anisotropy classes. First we prove that for every $C \in Ch_E$ which does not belong to Ch_\emptyset , $f(C)$ is entirely determined by the values of f on some finite set of chambers in a given Γ -stable apartment containing C . We start with the following result:

Proposition 5.5. *Assume E/F is tamely ramified. Let C be a chamber of X_E whose geometric realization is not contained in \mathcal{B}_F . Let A be a Γ -stable apartment of X_E containing C , let D be a wall of C and let C' be the other chamber of A*

admitting D as a wall. Assume that C' is not contained in the closure $\text{cl}(C \cup \gamma(C))$ and that D and $\gamma(D)$ are not contained in the same hyperplane of A . Let Ch_D be the set of chambers of X_E admitting D as a wall and distinct from C ; then $G_{F,der} \cap K_{C \cup \gamma(C)}$ acts transitively on Ch_D .

First we observe that since C and $\gamma(C)$ are both contained in the same half-plane delimited by the wall of A containing D (resp $\gamma(D)$), they are then both contained in the closure of $C' \cup \gamma(C')$. In particular, we have $K_{C' \cup \gamma(C'),E} \subset K_{C \cup \gamma(C),E}$.

Let T be the E -split maximal F -torus of G corresponding to A ; since $R(C)$ is not contained in \mathcal{B}_F , T is not F -split. Let g be an element of G_E such that $gTg^{-1} = T_0$; Γ then acts on the root system of G relative to T , which is $\text{Ad}(g)^{-1}\Phi$, and its action is nontrivial. For every $\alpha \in \Phi$, let $U_{\text{Ad}(g)^{-1}\alpha}$ be the root subgroup of G (relative to T) corresponding to $\text{Ad}(g)^{-1}\alpha$.

Let H be the hyperplane of A containing D , and let α be the element of Φ such that the root $\text{Ad}(g)^{-1}\alpha$ corresponds to the half-space \mathcal{S} of A delimited by H and containing C ; the group $U_{\text{Ad}(g)^{-1}\alpha,C} = U_{\text{Ad}(g)^{-1}\alpha} \cap K_{C,E}$ then acts transitively on Ch_D ; moreover, since $\gamma(H) \neq H$, \mathcal{S} contains both $\gamma(C)$ and $\gamma(C')$, hence $U_{\text{Ad}(g)^{-1}\alpha,C}$ fixes every element of $\gamma(Ch_D)$; we deduce from this that $\gamma(U_{\text{Ad}(g)^{-1}\alpha,C})$ fixes every element of Ch_D . Let now C'' be any element of Ch_D and let u be an element of the group $U_{\text{Ad}(g)^{-1}\alpha,C}$ such that $uC' = C''$; u (resp. $\gamma(u)$) then fixes both $\gamma(C')$ and $\gamma(C'')$ (resp. both C' and C'') and we obtain:

$$\gamma(u)uC' = u\gamma(u)C' = C''$$

and:

$$\gamma(u)u\gamma(C') = u\gamma(u)\gamma(C') = \gamma(C''),$$

We deduce from the above equalities that $h = u^{-1}\gamma(u^{-1})u\gamma(u)$ fixes both C' and $\gamma(C')$, hence belongs to $K_{C' \cup \gamma(C'),E}$. Moreover, since C' is a chamber, $K_{C' \cup \gamma(C'),E}$ is contained in an Iwahori subgroup of G_E , hence is pro-solvable, and since h is a product of unipotent elements of $K_{C' \cap \gamma(C'),E}$, it then belongs to the pro-unipotent radical $K_{C' \cup \gamma(C'),E}^0$ of $K_{C' \cup \gamma(C'),E} \subset K_{C \cup \gamma(C),E}$.

Moreover, we have $h\gamma(h) = 1$, hence h defines once again a 1-cocycle of $\Gamma = \{1, \gamma\}$ in $K_{C' \cup \gamma(C'),E}^0$. On the other hand, since E/F is tamely ramified, by [9, corollary 1], the cohomology set $H^1(\Gamma, K_{C' \cup \gamma(C'),E}^0)$ is trivial, hence there exists $h' \in K_{C' \cup \gamma(C'),E}^0$ such that $h = h'^{-1}\gamma(h')$, which implies:

$$u^{-1}\gamma(u^{-1})u\gamma(u)\gamma(h')^{-1}h' = 1.$$

Set $g' = u\gamma(u)\gamma(h')^{-1}$; we obtain $g' = \gamma(u)uh'^{-1} = \gamma(g')$, hence $g' \in G_{F,der}$, and $g'C' = C''$. Since this is true for every C'' , $G_{F,der} \cap K_{C \cup \gamma(C),F}$ acts transitively on Ch_D , as required. \square

Corollary 5.6. *Assume E/F is tamely ramified. Let A be a Γ -stable apartment of X_E , let Ch_A be the set of chambers of X_E contained in A and let f be an*

element of $\mathcal{H}(X_E)^{G_{F,der}}$. The restriction of f to Ch_A is entirely determined by the values of f on the chambers of Ch_A containing a facet of maximal dimension of the set A^Γ of Γ -stable elements of A . More precisely, if C is any chamber of Ch_A and C' is a chamber of Ch_A containing a facet of maximal dimension of A^Γ and whose combinatorial distance to C is the smallest possible, then $f(C)$ depends only on $f(C')$ and conversely.

Let C, C' be two elements of Ch_A ; assume C' contains a facet of maximal dimension of A^Γ . Let $(C_0 = C', C_1, \dots, C_r = C)$ be a minimal gallery between C' and C ; assume also that C' has been chosen in such a way that r is the smallest possible. For every i , let $D_i = C_{i-1} \cap C_i$; if D_i and $\gamma(D_i)$ are not contained in the same wall of A , by proposition 5.5 (applied to the chambers containing D_i) and the harmonicity condition, we have either $f(C_i) + qf(C_{i-1}) = 0$ or $qf(C_i) + f(C_{i-1}) = 0$, hence $f(C_i)$ is determined by $f(C_{i-1})$ and conversely. Hence if for every i , D_i satisfies that condition, by an obvious induction, we obtain that $f(C)$ is determined by $f(C')$ and conversely.

Assume now there exists some i such that D_i and $\gamma(D_i)$ are both contained in some wall H of A ; H is then Γ -stable. Let s_H be the reflection of A relative to H , or in other words the only simplicial automorphism of A fixing H pointwise; since H is Γ -stable, $\gamma \circ s_H \circ \gamma^{-1}$ is also such an automorphism, and must then be equal to s_H ; in other words, the action of γ on A commutes with s_H , from which we deduce that $s_H(C')$ contains a facet of maximal dimension of A^Γ . On the other hand, we have $s_H(C_{i-1}) = C_i$, hence $(s_H(C'), s_H(C_1), \dots, s_H(C_{i-2}), C_i, \dots, C_r = C)$ is a gallery (not necessarily minimal) between $s_H(C')$ and C of length $r - 1$; there must then exist a minimal gallery between them of length strictly smaller than r , which contradicts the minimality of r . Hence D_i and $\gamma(D_i)$ are never contained in the same hyperplane of A and the corollary is proved. \square

Now we prove that when G is not of type A_{2n} , the elements of $\mathcal{H}(X_E)^{G_{F,der}}$ are identically zero on most of the F -anisotropy classes of X_E (actually all but one, as we will see later with the help of proposition 5.11):

Proposition 5.7. *Assume E/F is tamely ramified, and G is not of type A_{2n} for any n . Let C be an element of Ch_E such that Σ_C is of cardinality $d - 1$ and not maximal as a set of strongly orthogonal elements of Φ^+ . Then for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$, $f(C) = 0$.*

Note first that, by lemma 4.4 and the following remark, the condition on Σ_C in fact already implies that G is not of type A_{2n} . This is also true for the second assertion of proposition 5.8.

Let f be any element of $\mathcal{H}(X_E)^{G_{F,der}}$, let A be a Γ -stable apartment of X_E containing C and let T be the E -split F -torus of G associated to A ; by eventually conjugating C by some element of G_F we can assume that the split component T_s of T is contained in T_0 , and even that T is contained in the F -split reductive subgroup L_0 of G defined as in proposition 4.5. Moreover, since Σ_C is of cardinality $d - 1$ and not maximal, there exists a unique $\alpha \in \Phi^+$ which is strongly

orthogonal to every element of Σ_C . The root subgroups U_α and $U_{-\alpha}$ are then normalized by T_0 and by every $U_{\pm\beta}$, $\beta \in \Sigma_C$, hence by L_0 .

Let $h \in L_0$ be such that $hT_0h^{-1} = T$; since α is orthogonal to every element of Σ_C , the root $Ad(h)\alpha$ of T does not depend on the choice of h . Let H_α be a wall of A corresponding to $Ad(h)\alpha$ and containing some facet of C , and let H'_α be the wall of A corresponding to the same root α , neighboring H_α and such that C is contained in the slice between them. Let D be a facet of maximal dimension, hence of dimension 1, of $A^\Gamma \subset A_0$, whose combinatorial distance to C is the smallest possible; D is then the unique edge of A^Γ whose vertices are contained respectively in H_α and H'_α . By corollary 5.6, $f(C)$ depends only on $f(C')$ for some chamber C' of A containing D , and conversely.

Let f_D be the concave function on Φ associated to D ; since α is not a linear combination of the elements of Σ_C , we must have $f_D(\alpha) + f_D(-\alpha) = \frac{1}{2}$, hence either $f_D(\alpha)$ or $f_D(-\alpha)$, say for example $f_D(\alpha)$, is an integer. Let D' be a facet of maximal dimension of H_α and let C'' be the unique chamber of A containing D' and whose remaining vertex is on the same side of H_α as H'_α ; we have $K_{D',F}/K_{C'',F} \subset K_{D',E}/K_{C'',E}$. If we prove that these two quotients are equal, then we obtain that $K_{D',F}$ acts transitively on the set of chambers containing D' ; if in addition we prove that every class of $K_{D,F}/K_{C,F}$ contains elements of $G_{F,der}$, we then obtain by $G_{F,der}$ -invariance and the harmonicity condition that the value of f on every such chamber is zero, and in particular that $f(C'') = 0$.

We thus prove that $K_{D',F}/K_{D',F}^0 = K_{D',E}/K_{D',E}^0$, from which the first part of our claim follows immediately. Since L_0 normalizes the root subgroup U_α of G associated to α and $K_D \cap L$ is a compact subgroup of L , we must have $hU_{\alpha,f_D(\alpha)}h^{-1} = U_{\alpha,f_D(\alpha)}$, and since $f_D(\alpha)$ is an integer, the quotient $U_{\alpha,f_D(\alpha)}/U_{\alpha,f_D(\alpha)+\frac{1}{2}}$ admits a system of representatives contained in $G_{F,der}$. Hence $U_{\alpha,f_D(\alpha)}$ is included in $K_{D',E}$, and $U_{\alpha,f_D(\alpha)+\frac{1}{2}} = U_{\alpha,f_D(\alpha)} \cap K_{D',E}^0$. On the other hand, by the same reasoning, we have $U_{-\alpha,-f_D(\alpha)} \subset K_{D',E}$ and $U_{-\alpha,-f_D(\alpha)+\frac{1}{2}} = U_{-\alpha,-f_D(\alpha)} \cap K_{D',E}^0$; hence the root subgroups of $K_{D',E}/K_{D',E}^0$ associated to both α and $-\alpha$ are contained in $K_{D',F}/K_{D',F}^0$, which is enough to prove that these two groups are equal. Moreover, at least q classes of $K_{D',F}/K_{C'',F}$ out of $q+1$ contain elements of $U_\alpha \subset G_{F,der}$, hence the quotient $K_{D',F} \cap G_{F,der}/K_{C'',F} \cap G_{F,der}$, whose cardinality divides $q+1$, must be isomorphic to $K_{D',F}/K_{C'',F}$ and the second part of the claim is proved.

Now if we choose D' in such a way that C'' is at minimal combinatorial distance from C'' among the chambers containing a facet of dimension 1 of A^Γ , by corollary 5.6, we then have $f(C'') = 0$, and then, also by the same corollary, $f(C) = 0$, which proves the proposition. \square

More generally, we have:

Proposition 5.8. • Assume E/F is tamely ramified, and G is not of type A_{2n} . Let C be an element of Ch_E which does not belong to Ch_\emptyset and let Σ_C be a subset of strongly orthogonal roots of Φ corresponding to the F -anisotropy class of C . Let Σ_C^\perp be the set of elements of Φ which are

strongly orthogonal to every element of Σ_C . Then Σ_C^\perp is a closed root subsystem of Φ .

- *Assume in addition that Σ_C and Σ_C^\perp are both nonempty and that Σ_C^\perp is of rank $d - \#(\Sigma_C)$. Then for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$, $f(C) = 0$.*

To prove that Σ_C^\perp is a closed root subsystem of Φ , we only need to prove that:

- for every $\alpha, \alpha' \in \Sigma_C^\perp$ such that $\alpha + \alpha' \in \Phi$, $\alpha + \alpha' \in \Sigma_C^\perp$;
- for every $\alpha \in \Sigma_C^\perp$, $-\alpha \in \Sigma_C^\perp$.

For every $\alpha \in \Sigma_C^\perp$, consider the reflection s_α associated to α . Since α is orthogonal to every element of Σ_C , s_α fixes Σ_C pointwise, which implies that Σ_C^\perp is stable by s_α , and in particular that it contains $s_\alpha(\alpha) = -\alpha$. Now let α, α' be two elements of Σ_C^\perp such that $\alpha + \alpha'$ is a root; since both of them are orthogonal to every element of Σ_C , then so is $\alpha + \alpha'$. Assume there exists $\beta \in \Sigma_C$ such that $\alpha + \alpha' + \beta$ is a root. Then β is orthogonal to both α and α' , which implies that α, α' and β are linearly independent; on the other hand, we deduce from lemma 4.3 that Φ is not simply-laced and $\alpha + \alpha'$ and β are both short, which also implies, since $\alpha + \alpha'$ and β are orthogonal, that $\alpha + \alpha' + \beta$ is long; the roots α, α' and β then generate a subsystem Φ' of Φ which is irreducible, not simply-laced and of rank 3, hence of type either B_3 or C_3 . Moreover, since $\alpha + \alpha'$ is short, either α or α' , say α , must be short.

In both cases below, the characters ε_i , $1 \leq i \leq d$, are respectively defined as in plates II and III of [4].

- Assume Φ' is of type B_3 . In a system of type B_d , the sum of two nonproportional short roots $\pm\varepsilon_i$ and $\pm\varepsilon_j$ is always a long root $\pm\varepsilon_i \pm \varepsilon_j$. Hence $\alpha + \beta$ is a root, which contradicts the fact that $\alpha \in \Sigma_C^\perp$.
- Assume Φ' is of type C_3 . In a system of type C_d , two strongly orthogonal short roots are of the form $\pm\varepsilon_i \pm \varepsilon_j$ and $\pm\varepsilon_k \pm \varepsilon_l$, with i, j, k, l being all distinct, which is obviously possible only if $d \geq 4$; hence α and β cannot be strongly orthogonal, which once again leads to a contradiction.

Hence such a β does not exist and $\alpha + \alpha' \in \Sigma_C^\perp$, which proves the first assertion of proposition 5.8.

Now we prove the second one. Assume first Σ_C^\perp is irreducible. Let A, D and f_D be defined as in the proof of proposition 5.7 and let D_1, \dots, D_{r+1} be the facets of D of dimension $r - 1$, with $r = d - \#(\Sigma_C)$ being the dimension of D . Let H_1, \dots, H_{r+1} be the hyperplanes of A respectively associated to the roots $\pm\alpha_1, \dots, \pm\alpha_{r+1}$ of Σ_C^\perp which respectively contain D_1, \dots, D_{r+1} ; the H_i are then actually walls of A , and if for every i , α_i is the one among $\pm\alpha_i$ which is oriented towards C , the set $\{\alpha_1, \dots, \alpha_{r+1}\}$ is an extended set of simple roots of Σ_C^\perp . If $\lambda_1, \dots, \lambda_{r+1}$ are the smallest positive integers such that $\lambda_1\alpha_1 + \dots + \lambda_{r+1}\alpha_{r+1} = 0$, we must have $\lambda_1 f_D(\alpha_1) + \dots + \lambda_{r+1} f_D(\alpha_{r+1}) = \frac{1}{2}$; on the other hand, if Σ_C^\perp is irreducible and not of type A_{2n} for any n , by [4, §I, proposition 31], the sum of the λ_i is even, which implies that at least one of the $f_D(\alpha_i)$ must be an integer, and we

finish the proof in a similar way as in proposition 5.7. When Σ_C^\perp is reducible and has no irreducible component of type A_{2n} for any n , considering each irreducible component of Σ_C^\perp separately, the proof is similar.

Now we check that Σ_C^\perp cannot possibly have any irreducible component of type A_{2n} . Assume it admits such a component. Then the set $\Phi_C = \Sigma_C \cup -\Sigma_C \cup \Sigma_C^\perp$ is a proper closed root subsystem of Φ of rank d admitting at least one component of type A_1 since Σ_C is nonempty, and at least one component of type A_{2n} for some n , which implies in particular that $d \geq 3$. Assume first that Φ_C is a parahoric subsystem of Φ , or in other words the subsystem generated by $\Delta' - \{\alpha\}$, where Δ' is an extended set of simple roots of Φ and α is a nonspecial element of Δ' ; its Dynkin diagram is then the extended Dynkin diagram of Φ with the vertex corresponding to α removed. By examining the diagrams of the various possible parahoric subsystems of root systems of every type, we see that Φ_C can possibly have the required irreducible components only when Φ is of type E_8 , $r = 7$ and Σ_C^\perp is of type $A_2 \times A_5$, which implies that Φ_C is of type $A_1 \times A_2 \times A_5$. On the other hand, if Φ is of type E_8 and Σ_C is a singleton, it is easy to check that Σ_C^\perp must be of type E_7 ; we thus obtain a contradiction.

Now we look at the general case. By [13, theorem 5.5] and an obvious induction, we obtain a tower of root systems $\Phi = \Phi_0 \supset \Phi_1 \supset \dots \Phi_s = \Phi_C$ such that Φ_i is a parahoric subsystem of Φ_{i-1} for every i and that Φ_s admits the required irreducible components. We deduce from the above discussion that if Φ_{s-1} is irreducible, it must be of type E_8 , which, since E_8 is not contained in any other root system of rank 8 (A_8 and D_8 are both strictly contained in E_8 , as well as the systems of long roots of B_8 and C_8 , which are respectively D_8 and A_1^8), implies $s = 1$, we are then reduced to the previous case. Assume now Φ_{s-1} is reducible. Then in order for Φ_C to admit any component of type A_{2n} , there must exist an i such that Φ_i admits such an irreducible component and Φ_{i-1} does not. The possible cases are, apart from the one which is already ruled out:

- Φ_{i-1} admits an irreducible component of type E_6 and that component breaks into three components of type A_2 in Φ_i . According to the table on page 29 of [7], every vertex of the Dynkin diagram of a root system of type A_n is special; we deduce from this that such a root system has no proper subsystems of the same rank. This implies that no component of type A_1 can arise in Φ_s from these three components, hence the components forming $\Sigma_C \cup -\Sigma_C$ must come from the other components of Φ_{i-1} . But then the whole component of type E_6 of Φ_{i-1} is contained in $\Sigma_C^\perp \subset \Phi_s$, which contradicts the fact that it is already not contained in Φ_i .
- Φ_{i-1} admits a component of type E_7 which breaks into a system of type $A_2 \times A_5$ in Φ_i . For the same reason as above, Φ_{i-1} must admit components distinct from that component of type E_7 and containing the whole $\Sigma_C \cup -\Sigma_C$, and we reach the same contradiction.

- Φ_{i-1} is of type E_8 and Φ_i is of type $E_6 \times A_2$. Since the only possible way for Φ_s to have any component of type A_1 is that the component of type E_6 breaks into $A_1 \times A_5$, we are reduced to a previous case.
- Φ_{i-1} is of type E_8 and Φ_i is of type $A_4 \times A_4$. There is no way that Φ_s can ever have any component of type A_1 , since such a component should come from one of these two components of type A_4 and we already know that it is impossible.
- Φ_{i-1} is of type E_8 and Φ_i is of type A_8 . Same as above.
- Φ_{i-1} is of type F_4 and Φ_i is of type $A_2 \times A_2$. Same as above.
- Φ_{i-1} is of type G_2 and Φ_i is of type A_2 . This case is ruled out by the fact that we must have $d \geq 3$.

Since we always reach a contradiction, Σ_C^\perp cannot have any irreducible component of type A_{2n} and the proposition is proved. \square

Note that in the course of the above proof, we have proved the following lemma which will be useful later:

Lemma 5.9. *Let Σ be a subset of strongly orthogonal elements of Φ . Assume at least two elements of Σ are short. Then G is of type C_d , with $d \geq 4$, and these two short elements of Σ are of the form $\pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_k \pm \varepsilon_l$, with i, j, k, l being all distinct.*

The following proposition allows us to eliminate more F -anisotropy classes from the support of the harmonic cochains:

Proposition 5.10. *Assume E/F is tamely ramified. Let C, C' be two adjacent chambers of X_E , and let D be the wall separating them. Let A (resp. A') be a Γ -stable apartment of X_E containing C (resp. C') and let T (resp. T') be the corresponding E -split maximal F -torus of G . Let Σ (resp. Σ') be a subset of strongly orthogonal roots of Φ corresponding to the F -anisotropy class of T (resp. T'); assume that:*

- **(C1)** *there exist $\alpha \in \Sigma'$ and $\beta \in \Phi$ such that β is orthogonal to every element of Σ' except α and that $\langle \alpha, \beta^\vee \rangle$ is odd;*
- $\Sigma' = \Sigma \cup \{\alpha\}$.

Let Ch_D be the set of chambers of X_E containing D and distinct from both C and the other chamber C'' containing D and contained in A . Then $G_{F,der} \cap K_{C \cup \gamma(C)}$ acts transitively on Ch_D .

By eventually conjugating C and C' by some element of G_F we can assume that the F -split component of T' is contained in T_0 . Let g (resp. g') be an element of G_E such that $gT_0g^{-1} = T$ (resp. $g'T_0g'^{-1} = T'$); define Σ_T and L_0 as in proposition 4.5 and $\Sigma_{T'}$ and L'_0 in a similar way (relative to T' instead of T), and set $L = gL_0g^{-1}$ and $L' = g'L'_0g'^{-1}$; since, by lemma 4.6, L and L' are both F -split, we obtain that L is a G_F -conjugate of some subgroup of L' , and by multiplying g' by a suitable element of the normalizer of T_0 in G_F , we actually

obtain $L \subset L'$. The roots corresponding to the hyperplane of A' containing D are then $\pm\alpha$; for every one-parameter subgroup ξ of T_0 orthogonal to every element of Σ , $\xi(\mathcal{O}_F)$ then stabilizes Ch_D globally. Moreover, by **(C1)**, there exists a one-parameter subgroup ξ in X^\vee which is orthogonal to every element of Σ and such that $\langle \alpha, \xi \rangle$ is odd, and by adding to ξ a suitable multiple of α^\vee we can assume that $\langle \alpha, \xi \rangle = 1$. Hence $\alpha \circ \xi$ is the identity on F^* , and in particular its restriction to Ch_F is surjective, which implies that $\xi(\mathcal{O}_F^*)$, which is contained in $G_{F,der} \cup K_{C' \cup \gamma(C')}$, acts transitively on Ch_D . \square

Now we consider the F -anisotropy classes which are not covered by the previous induction. Actually we prove that there is no such class when G is of type A_{2n} , and exactly one when G is of any other type:

Proposition 5.11. (1) *Assume Φ is not of type A_{2n} for any n . There exists a subset Σ_a of Φ , unique up to conjugation by an element of the Weyl group of Φ , satisfying the following properties:*

- *for every $\alpha, \beta \in \Sigma_a$, α and β are strongly orthogonal, and Σ_a is maximal for that property;*
- *Σ_a does not satisfy **(C1)**.*

- (2) *With the same hypothese, Σ_a is also maximal as a set of orthogonal roots of Φ .*
- (3) *With the same hypothese once again, every subset of strongly orthogonal elements of Φ which does not satisfy **(C1)** is a conjugate of some subset of Σ_a .*
- (4) *Assume now Φ is of type A_{2n} for some n . Then every nonempty subset of strongly orthogonal roots of Φ satisfies **(C1)**. In particular, a subset Σ_a of Σ defined as above cannot exist.*

First consider the case A_{2n} ; we prove (4) by induction on n . When $n = 1$, no two roots of Φ are orthogonal to each other, which implies that every nonempty subset of orthogonal roots of Φ is a singleton; on the other hand, if α, β are any two nonproportional roots of Φ , we have $\langle \alpha, \beta^\vee \rangle = \pm 1$, hence $\{\alpha\}$ satisfies **(C1)**. Assume now $n > 1$, and let Σ be any subset of strongly orthogonal elements of Φ . Let α be any element of Σ ; the subsystem Φ' of the elements of Φ which are orthogonal to α is then of type A_{2n-2} , and admits $\Sigma - \{\alpha\}$ as a subset of strongly orthogonal elements. If $\Sigma - \{\alpha\}$ is empty, then taking as β any element of Φ which is neither proportional nor orthogonal to α , we see that $\Sigma = \{\alpha\}$ satisfies **(C1)**. Now assume $\Sigma - \{\alpha\}$ is nonempty. By the induction hypothesis, $\Sigma - \{\alpha\}$ must satisfy **(C1)** as a subset of Φ' . Let then $\alpha' \in \Sigma - \{\alpha\}$ and $\beta \in \Phi'$ be such that β is orthogonal to every element of $\Sigma - \{\alpha, \alpha'\}$ and $\langle \alpha, \beta^\vee \rangle$ is odd; by definition of Φ' , β is also orthogonal to α . Hence Σ satisfies **(C1)** and (4) is proved.

Assume now Φ is not of type A_{2n} ; we first prove the existence of Σ_a . First consider the cases covered by lemma 4.4, or in other words assume that there exists $w \in W$ such that $w(\alpha) = -\alpha$ for every $\alpha \in \Phi$; by lemma 4.4, there

exists then a subset Σ_a of d strongly orthogonal elements of Φ ; such a subset is necessarily maximal, and for every $\alpha \in \Sigma_a$, the only elements of Φ which are strongly orthogonal to every element of $\Sigma_a - \{\alpha\}$ are $\pm\alpha$, and since $\langle \alpha, \alpha^\vee \rangle = 2$ is even, Σ_a does not satisfy **(C1)**, as required.

Now we consider the remaining cases. Using the same algorithm as for lemma 4.4 (taking the highest root α_0 of Φ^+ and then considering the subsystem of the elements of Φ which are strongly orthogonal to α_0), we also obtain a maximal subset Σ_a of strongly orthogonal roots of Φ , but this time, since $w = \prod_{\beta \in \Sigma_a} s_\beta$ cannot satisfy $w(\alpha) = -\alpha$ for every $\alpha \in \Phi$, by lemma 4.4, Σ_a contains strictly less than d elements; we claim that for every $\alpha \in \Sigma_a$, the only elements of Φ which are strongly orthogonal to every element of $\Sigma_a - \{\alpha\}$ are $\pm\alpha$ once again.

Remember that the root systems we are considering here are the types A_{2n-1} for some $n > 1$ (A_{2n} being already ruled out), D_{2n+1} for some n and E_6 : since all these systems are simply-laced, by [4, §1, 10, proposition 1], two elements of Σ_a are always conjugates of each other, which implies that we only have to prove the claim for one given $\alpha \in \Sigma_a$. By eventually conjugating Σ_a , we can always assume it contains α_0 . In the sequel, the simple roots $\alpha_1, \dots, \alpha_d$ of Φ^+ are numbered as in [4, plates I to IX].

- Assume first Φ is of type A_{2n-1} , $n \geq 2$. The subsystem Φ' of the elements of Φ which are strongly orthogonal to α_0 is then generated by the α_i , $2 \leq i \leq 2n-2$, hence of type A_{2n-3} . On the other hand, if α' is an element of Σ_a distinct from α , it is contained in Φ' , and if the assertion is true for Φ' , $\Sigma_a - \{\alpha_0\}$ and α' , then it is also true for Φ , Σ_a and α' ; we are then reduced to the case of type A_{2n-3} . By an obvious induction, after a finite number of such reductions we reach the case of a system of type A_1 , and in that case, $\Sigma_a = \{\alpha_0\}$ obviously satisfies the required condition.
- Assume now Φ is of type D_{2n+1} , $n \geq 2$. The subsystem of the elements of Φ which are strongly orthogonal to α_0 is then generated by the α_i , $i \neq 2$, hence of type $A_1 \times D_{2n-1}$, the component of type A_1 being $\{\pm\alpha_1\}$. By eventually conjugating Σ_a by the reflection s_{α_1} , we may assume it contains α_1 as well as α_0 , and by a similar reasoning as above (considering $\Sigma_a - \{\alpha_0, \alpha_1\}$ instead of $\Sigma_a - \{\alpha_0\}$), we are reduced to the case of type D_{2n-1} ; after a finite number of such reductions we reach the case of a system of type $D_3 = A_3$, which is an already known case.
- Assume finally Φ is of type E_6 . The subsystem of the elements of Φ which are strongly orthogonal to α_0 is then generated by the α_i , $i \neq 2$, hence of type A_5 , and we are once again reduced to an already known case.

Now we prove the unicity of Σ_a (up to conjugation) by induction on d , the case $d = 1$ being obvious. Let Σ be any subset of Φ satisfying the conditions of the proposition. Assume Σ contains at least one long root (recall that by convention every root of a simply-laced system is considered long); by eventually conjugating Σ , we can assume that root is α_0 , and if Ψ is the subsystem of the elements

of Φ which are strongly orthogonal to α_0 , $\Sigma - \{\alpha_0\}$ satisfies the conditions of the proposition as a subset of Ψ , hence by induction hypothesis $\Sigma - \{\alpha_0\}$ and $\Sigma_a - \{\alpha_0\}$ are conjugated by an element w of the Weyl group W_Ψ of Ψ . Since α_0 is orthogonal to every element of Ψ , it is fixed by W_Ψ , hence Σ and Σ_a are conjugated by w .

Assume now Φ is not simply-laced and Σ contains only short roots. We now examine the different cases:

- Assume first Φ is of type G_2 . Then no two roots of Φ are orthogonal, hence Σ must be a singleton $\{\alpha\}$. Since there are long roots in Φ which are orthogonal to α , hence strongly orthogonal by lemma 4.3, Σ cannot be maximal.
- Assume now Φ is of type C_d . Let $\varepsilon_1, \dots, \varepsilon_d$ be defined as in [4, plate III]. We have already seen (lemma 5.9) that when Φ is of type C_d and Σ contains only short roots, these roots must be of the form $\pm\varepsilon_i \pm \varepsilon_j$ with no two indices being identical; on the other hand, every possible index must show up in some $\pm\varepsilon_i \pm \varepsilon_j$, since if some index k does not, the long root $2\varepsilon_k$ is strongly orthogonal to every element of Σ , which contradicts the maximality of Σ . Hence $d = 2n$ is even and the only possible Σ is, up to conjugation: $\Sigma = \{\varepsilon_1 + \varepsilon_2, \dots, \varepsilon_{2n-1} + \varepsilon_{2n}\}$. On the other hand, the long root $\beta = 2\varepsilon_1$ is orthogonal to every element of Σ but $\alpha = \varepsilon_1 + \varepsilon_2$, and we have $\langle \alpha, \beta^\vee \rangle = 1$, which contradicts the fact that Σ must not satisfy **(C1)**.
- Assume now Φ is of type either B_d or F_4 . In both these cases, it is easy to check that no two orthogonal short roots are strongly orthogonal, hence Σ must be a singleton. Let Φ' be a subsystem of type $B_2 = C_2$ of Φ containing Σ ; according to the previous case, Σ satisfies **(C1)** as a subset of Φ' , hence also as a subset of Φ and we reach a contradiction once again.

In all the above cases, either Σ is a conjugate of Σ_a or we have reached a contradiction. Hence (1) is proved.

Now we prove (2). Assume there exists $\alpha \in \Phi$ which is orthogonal to every element of Σ_a . Then at least one element of Σ_a is orthogonal but not strongly orthogonal to α , which implies by lemma 4.3 that Φ is not simply-laced. On the other hand, Σ_a is then of cardinality strictly smaller than d , which by lemma 4.4 and the following remark is possible only if Φ is of type A_d , with $d > 1$ odd, D_d , with d odd, or E_6 , hence simply-laced. We thus reach a contradiction, hence α cannot exist and (2) is proved.

Now we prove (3). When Φ is simply-laced, we deduce from (1) that every maximal subset of strongly orthogonal roots of Φ which does not satisfy **(C1)** is conjugated to Σ_a , and (3) follows immediately. When Φ is of type G_2 , it is easy to check that every maximal subset of strongly orthogonal roots of Φ must always contain one long root and one short root, which also implies (3). When Φ is of type B_{2n+1} for some n , every subset of strongly orthogonal elements of

Φ contains at most one short root (since in a system of type B_d , the sum of two nonproportional short roots is always a long root), and at most $2n$ long roots (since all of these long roots must be contained in the subsystem of the long roots of Φ , which is of type D_{2n+1} and, as we have already seen, does not contain any subset of strongly orthogonal elements of cardinality $2n+1$); using the induction of lemma 4.4 once again, we easily see that such a subset must also be contained in a conjugate of Σ_a ; the assertion (3) follows immediately in that case too.

It remains to consider the cases B_{2n} , C_d and F_4 . In all these cases, Σ_a contains only long roots: this is easy to check by examining the subsystem Φ_l of the long roots of Φ , which is of type respectively D_{2n} , A_1^d and D_4 ; in all three cases, Φ_l contains a subset of d strongly orthogonal roots which does not satisfy (C1), and such a subset must then be a conjugate of Σ_a in Φ_l , hence also in Φ . If Σ contains only long roots, by replacing Φ by Φ_l , we are reduced to the simply-laced case. Assume now Σ contains at least one short root α ; we prove by induction on the number of short roots it contains that it must satisfy (C1). By induction hypothesis, if Φ' is the subsystem of the elements of Φ which are strongly orthogonal to α , $\Sigma - \{\alpha\}$ either satisfies (C1) as a subset of Φ' or is contained in some conjugate of Σ_a that by conjugating Σ we may assume to be Σ_a itself. In the first case, by the same argument as in the case A_{2n} , Σ must satisfy (C1) as a subset of Φ . In the second case, since Σ_a is of cardinality d , α is a linear combination of the elements of Σ_a , which is possible only if there exist $\beta_1, \beta_2 \in \Sigma_a$ such that $\alpha = \frac{1}{2}(\pm\beta_1 \pm \beta_2)$. We then have:

$$\langle \alpha, \beta_1^\vee \rangle = \pm \langle \alpha, \beta_2^\vee \rangle = \pm 1,$$

which proves at the same time that β_1 and β_2 do not belong to Σ and that Σ satisfies (C1). Hence (3) is proved. \square

Corollary 5.12. *Assume Φ is not of type A_{2n} for any n , E/F is tamely ramified and Σ' is a subset of strongly orthogonal roots of Φ which either is not maximal or satisfies (C1). Then for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$ and every $C \in Ch_E$ of anisotropy class Σ' , $f(C) = 0$.*

Assume first Σ_a is of cardinality d . If Σ' is a conjugate of some subset of Σ_a , then the set Σ'^\perp of elements of Φ which are strongly orthogonal to Σ' contains some conjugate of $\Sigma_a - \Sigma'$, hence is of dimension $d - \#(\Sigma')$ and we can just apply proposition 5.8 if Σ' is nonempty, and corollary 5.3 if Σ' is empty. Assume now Σ' is not a conjugate of any subset of Σ_a . By proposition 5.11(3), Σ' satisfies (C1) and we can proceed by induction. Let Σ, C, C' be defined as in proposition 5.10; if we assume $f(C) = 0$, by proposition 5.10 we have $f(C') = 0$ as well. By proposition 5.11(3), either Σ is a conjugate of some subset of Σ_a , in which case we have $f(C') = 0$ by the previous case, or Σ satisfies (C1), in which case we can just iterate the process; since Σ' is finite, after a finite number of steps we reach a situation where Σ is conjugated to a subset (eventually empty) of Σ_a , hence by the previous case $f(C) = 0$, and by proposition 5.10 and an obvious induction,

we must have $f(C') = 0$. The fact that f is then zero on the whole anisotropy class Σ' of Ch_E follows from corollary 5.6.

Assume now Σ_a is of cardinality smaller than d . Then Φ is simply-laced, hence, as we have seen during the proof of proposition 5.11, Σ' is always a conjugate of a subset of Σ_a . Now we examine the different cases:

- Assume Φ is of type A_{2n-1} . It is easy to check that for every $\alpha \in \Phi$, the subsystem of elements of Φ which are orthogonal to α is of type A_{2n-3} ; we deduce from this that every proper subset of Σ_a , and more generally every nonmaximal subset of strongly orthogonal roots of Φ , is contained in a subsystem of Φ of type A_{2n-3} , hence also in a subsystem of Φ of type A_{2n-2} ; by proposition 5.11(4), Σ' then satisfies **(C1)**. We thus can apply proposition 5.10 and an easy induction to get the desired result.
- Assume Φ is of type D_{2n+1} , and, the ε_i being defined as in [4, plate IV], set $\Sigma_a = \{\varepsilon_1 \pm \varepsilon_2, \dots, \varepsilon_{2n-1} \pm \varepsilon_{2n}\}$. It is easy to check (details are left to the reader) that the set of W -conjugacy classes of sets of strongly orthogonal elements of Φ admits as a set of representatives the set of subsets $\{\Sigma_{i,j} | 0 \leq i \leq j \leq n\}$, with $\Sigma_{i,j} = \{\varepsilon_1 \pm \varepsilon_2, \dots, \varepsilon_{2i-1} \pm \varepsilon_{2i}, \varepsilon_{2i+1} + \varepsilon_{2i+2}, \dots, \varepsilon_{2j-1} + \varepsilon_{2j}\}$; in particular, $\Sigma_{n,n} = \Sigma_a$. When $i < j$, setting for example $\alpha = \varepsilon_{2j-1} + \varepsilon_{2j}$ and $\beta = \varepsilon_{2j} + \varepsilon_{2j+1}$, we see that $\Sigma_{i,j}$ satisfies **(C1)**. However, this is not true for the $\Sigma_{i,i}$, $0 \leq i \leq n-1$, and we must then deal with them first. For every $i < n$, $\Sigma_{i,i}^\perp$ is a subsystem of type $D_{2(n-i)+1}$ of Φ , namely the set of roots which are linear combinations of the ε_j , $2i+1 \leq j \leq 2n+1$; its rank is then equal to $d - \#(\Sigma_{i,i})$, and we can then apply proposition 5.8 (or corollary 5.3 if $i = 0$) to obtain that $f(C) = 0$ in these cases. The cases $\Sigma_{i,j}$, $i < j$, then follow from the cases $\Sigma_{i,i}$ by proposition 5.10 and an easy induction.
- Assume Φ is of type E_6 ; Σ_a is then contained in a Levi subsystem Φ' of type D_4 of Φ , hence also in a Levi subsystem Φ'' of type D_5 ; we can thus define subsets $\Sigma_{i,j}$, $0 \leq i \leq j \leq 2$, of that last subsystem in a similar way as in the previous proposition, and we can even assume they are contained in Φ' . Moreover, if we assume that Φ' (resp. Φ'') is generated by the elements $\alpha_2, \dots, \alpha_5$ (resp. $\alpha_1, \dots, \alpha_5$) of Δ (the simple roots being numbered as in [4, plate V]), the elements of W corresponding to the order 3 automorphisms of the extended Dynkin diagram of Φ act on Φ' by automorphisms of order 3 of its Dynkin diagram, and in particular permute the subsets $\{\alpha_2, \alpha_3\}$, $\{\alpha_2, \alpha_5\}$ and $\{\alpha_3, \alpha_5\}$ of Φ ; we deduce from this that the sets $\Sigma_{1,1} = \{\alpha_2, \alpha_5\}$ and $\Sigma_{0,2} = \{\alpha_3, \alpha_5\}$ belong to the same conjugacy class of sets of strongly orthogonal elements of Φ . By proposition 5.10 and the previous induction applied to $\Sigma_{0,0} \rightarrow \Sigma_{0,1} \rightarrow \Sigma_{0,2}$ and then to $\Sigma_{1,1} \mapsto \Sigma_{1,2}$, we obtain the desired result.

The corollary is then proved. \square

Proposition 5.13. *In the various cases, the sets Σ_a are, up to conjugation, the following ones:*

- when G is of type A_{2n-1} , $\Sigma_a = \{-\varepsilon_1 + \varepsilon_{2n}, -\varepsilon_2 + \varepsilon_{2n-1}, \dots, -\varepsilon_n + \varepsilon_{n+1}\}$;
- when G is of type B_{2n} , $\Sigma_a = \{-\varepsilon_1 \pm \varepsilon_2, -\varepsilon_3 \pm \varepsilon_4, \dots, -\varepsilon_{2n-1} \pm \varepsilon_{2n}\}$;
- when G is of type B_{2n+1} , $\Sigma_a = \{-\varepsilon_1 \pm \varepsilon_2, -\varepsilon_3 \pm \varepsilon_4, \dots, -\varepsilon_{2n-1} \pm \varepsilon_{2n}, -\varepsilon_{2n+1}\}$;
- when G is of type C_d , $\Sigma_a = \{-2\varepsilon_1, \dots, -2\varepsilon_d\}$;
- when G is of type D_d , with d being either $2n$ or $2n+1$, $\Sigma_a = \{-\varepsilon_1 \pm \varepsilon_2, -\varepsilon_3 \pm \varepsilon_4, \dots, -\varepsilon_{2n-1} \pm \varepsilon_{2n}\}$;
- when G is of type E_6 , $\Sigma_a = \{-\alpha_0, -\alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, -\alpha_3 - \alpha_4 - \alpha_5, -\alpha_4\}$;
- when G is of type E_7 , $\Sigma_a = \{-\alpha_0, -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7, -\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5, -\alpha_2, -\alpha_3, -\alpha_5, -\alpha_7\}$;
- when G is of type E_8 , $\Sigma_a = \{-\alpha_0, -2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7, -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7, -\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5, -\alpha_2, -\alpha_3, -\alpha_5, -\alpha_7\}$;
- when G is of type F_4 , $\Sigma_a = \{-\alpha_0, -\alpha_2 - 2\alpha_3 - 2\alpha_4, -\alpha_2 - 2\alpha_3, -\alpha_2\}$;
- when G is of type G_2 , $\Sigma_a = \{-\alpha_0, -\alpha_1\}$.

The above sets Σ_a are simply the ones we obtain by applying the algorithm of lemma 4.4. Details are left to the reader. \square

Note that for convenience (to be able to make the best possible use of lemma 6.17), we may want in the sequel to use representatives of Σ_a which contain as many negatives of simple roots as possible, and we thus obtain:

Proposition 5.14. *In the following cases, these alternative Σ_a are also valid choices:*

- when G is of type A_{2n-1} , $\Sigma_a = \{-\alpha_1, -\alpha_3, \dots, -\alpha_{2n-1}\}$;
- when G is of type D_{2n+1} , $\Sigma_a = \{-\varepsilon_2 \pm \varepsilon_3, -\varepsilon_4 \pm \varepsilon_5, \dots, -\varepsilon_{2n} \pm \varepsilon_{2n+1}\}$;
- when G is of type E_6 , $\Sigma_a = \{\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, -\alpha_2, -\alpha_3, -\alpha_5\}$.

Checking that these sets are also valid representatives of Σ_a in their respective cases is straightforward, details are left to the reader. In the other cases, the representative of Σ_a we pick up is still the one given by proposition 5.13.

In particular, we have proved the following result:

Proposition 5.15. *It is possible to choose Σ_a in such a way that it is contained in a standard Levi subsystem Φ' of rank $\#(\Sigma_a)$ of Φ and that every one of its elements is the negative of the sum of an odd number of simple roots of Φ^+ (counted with multiplicities).*

Checking that the condition of proposition 5.15 is satisfied by the sets Σ_a given by proposition 5.14 in the cases covered by that proposition and by proposition 5.13 in the other cases is straightforward. \square

The last three results of this section are three more corollaries to proposition 5.11.

Let Ch_a be the subset of chambers of X_E of anisotropy class Σ_a , and let Ch_a^0 be the subset of the elements of Ch_a containing a Γ -fixed facet of maximal dimension of any Γ -stable apartment containing them.

Corollary 5.16. *Assume Φ is not of type A_{2n} for any n and E/F is tamely ramified. Let f be an element of $\mathcal{H}(X_E)^{G_{F,der}}$; the support of f is contained in Ch_a , and f is entirely determined by its values on Ch_a^0 .*

By proposition 5.11(3), every subset of strongly orthogonal roots of Φ which is not a conjugate of Σ_a either is not maximal or satisfies **(C1)**; the corollary then follows from corollaries 5.6 and 5.12. \square

In the case of groups of type A_{2n} , our induction actually works on the whole set Ch_E and we obtain:

Corollary 5.17. *Assume Φ is of type A_{2n} for some n and E/F is tamely ramified. Let f be an element of $\mathcal{H}(X_E)^{G_{F,der}}$; f is then entirely determined by its value on some given element of Ch_c . In particular, theorem 1.2 holds for groups of type A_{2n} .*

By proposition 5.11(4), every subset of strongly orthogonal roots of Φ satisfies **(C1)**; by corollary 5.6, proposition 5.10 and an easy induction, f is then entirely determined by its values on the set Ch_\emptyset of chambers of X_E whose geometric realization is contained in \mathcal{B}_F . On the other hand, by proposition 5.2, Ch_c is the only $G_{F,der}$ -orbit of chambers satisfying that condition and on which f can be nonzero, hence f is entirely determined by its values on Ch_c . In particular, $\mathcal{H}(X_E)^{G_{F,der}}$ is of dimension at most 1. As in [5], theorem 1.2 follows. \square

Note that we did not need to determine precisely the support of the elements of $\mathcal{H}(X_E)^{G_{F,der}}$ to prove the above corollary, so we do it now.

For every apartment A of X_E , we denote by Ch_A the set of chambers of A .

Corollary 5.18. *Assume Φ is of type A_{2n} for some n and E/F is tamely ramified. Then assuming $\mathcal{H}(X_E)^{G_{F,der}}$ contains nonzero elements, their support is the union of Ch_c and of the Ch_A , with A being a Γ -stable apartment of X_E whose geometric realization is not contained in \mathcal{B}_F and such that every facet of maximal dimension of A^Γ is a facet of some element of Ch_c .*

Let A' be any Γ -stable apartment of X_E , and let Σ' be a set of strongly orthogonal roots of Φ corresponding to the F -anisotropy class of the E -split maximal torus associated to A' . By proposition 5.11(4), every nonempty subset of Σ' satisfies **(C1)**. Let D be a facet of maximal dimension of A'^Γ ; we prove by induction on the cardinality of Σ' that for every nonzero $f \in \mathcal{H}(X_E)^{G_{F,der}}$, assuming such an f actually exists, f is nonzero on the set of chambers of A' containing D only if D is contained in some chamber of Ch_c , and f is then constant on that set. Let Σ , A , C , C' and C'' be defined as in proposition 5.10 relatively to Σ' and A' ; by that proposition and the harmonicity condition, $f(C') = 0$ if and only if

$f(C) + f(C'') = 0$, and we have:

$$f(C') = \frac{f(C) + f(C'')}{1 - q}.$$

On the other hand, if Σ' is a singleton, then C and C'' are two adjacent chambers in Ch_\emptyset , hence by definition of Ch_c , one of them can belong to Ch_c only if the wall separating them, which is D , is such that $R(D)$ is not contained in any wall of \mathcal{B}_F , and in such a case, $R(C)$ and $R(C'')$ are contained in the same chamber of \mathcal{B}_F ; by proposition 5.2, if, say, C belongs to Ch_c , then $C'' \notin Ch_c$. Hence by corollary 5.3, we have $f(C) + f(C'') \neq 0$, which implies $f(C') \neq 0$, if and only if D is contained in some chamber of Ch_c . On the other hand, the value of f on C' is then always equal to the constant value of f on Ch_c multiplied by $\frac{1}{1-q}$, hence nonzero. The fact that f is then nonzero on the whole set of chambers of A' is a consequence of corollary 5.6.

Assume now Σ' contains at least two elements. By induction hypothesis, we have $f(C) = f(C'')$, and they are nonzero if and only if both C and C'' contain a facet D' of maximal dimension of A^Γ contained in some chamber of Ch_c . Hence if $f(C') \neq 0$, D must be contained in Ch_c and we have $f(C') = \frac{2}{1-q}f(C)$. Conversely, if D is contained in some chamber of Ch_c , then it is contained in some D' satisfying the same condition and $f(C')$ is then nonzero. As in the previous case, we use corollary 5.6 to obtain that f is then nonzero on the whole set of chambers of A' . \square

6. THE SPHERICAL PART

In this section, we prove theorem 1.2 when Φ is not of type A_{2n} for any n . From now on until the end of the paper, we assume that E/F is tamely ramified.

Let Σ_a be a subset of strongly orthogonal roots of Φ satisfying the conditions of proposition 5.11, let \mathcal{A} be a Γ -stable apartment of \mathcal{B}_E whose associated torus is of F -anisotropy class Σ_a , let T be the E -split maximal torus associated to \mathcal{A} , and let D be a facet of X_E whose geometric realization is a facet of maximal dimension of \mathcal{A}^Γ ; we denote by Ch_D the set of chambers of X_E containing D . First we prove that the elements of $\mathcal{H}(X_E)^{G_{F,der}}$ are entirely determined by their restrictions to Ch_D for a suitably chosen D , then we prove that the space of restrictions of the elements of $\mathcal{H}(X_E)^{G_{der,F}}$ to Ch_D is of dimension at most 1.

To achieve that, we will continue to restrict our harmonic cochains to smaller sets. The general strategy is the following one: starting with the whole set Ch_E , we successively prove that we only have to consider the following subsets:

- the subset Ch_D of the elements of Ch_E which contain D ;
- the subset $Ch_{D,a}$ of the elements of Ch_D whose F -anisotropy class is (up to conjugation) Σ_a ;

- the subset $Ch_{D,a,L}$ of the elements of $Ch_{D,a}$ contained in some Γ -stable apartment A of X_E whose associated torus is contained in some given reductive subgroup L of G (namely, the one of proposition 4.5);
- the subset Ch_{D,a,L,C_0} of the elements of $Ch_{D,a,L}$ of the form uC_0 for some given Γ -fixed chamber C_0 of X_E containing D , where u is a product of elements of the root subgroups of L which correspond to elements of Σ_a .

Finally, we compute explicitly the restrictions to Ch_{D,a,L,C_0} of our harmonic cochains; by proposition 6.13, that set happens to be in 1-1 correspondence with some cohomology group which is easier to study.

6.1. Some preliminary results. We choose D arbitrarily for the moment. First we prove the following results:

Proposition 6.1. *Assume D is a single vertex x ; x is then a special vertex of X_E .*

(See section 2 for the definition of a special vertex.)

By eventually conjugating it by some element of $G_{F,der}$ we can always assume that $x \in \mathcal{A}_0$. The above statement can then be rewritten in terms of concave functions the following way: let Σ be a set of strongly orthogonal roots of Φ conjugated to Σ_a . Assume the cardinality of Σ is equal to the rank d of Φ and let f be a concave function from Φ to $\frac{1}{2}\mathbb{Z}$ such that $f(\alpha) \in \mathbb{Z} + \frac{1}{2}$ for every $\alpha \in \pm\Sigma$ (this property corresponds to the fact that D is a Γ -fixed facet of maximal dimension of an apartment of F -anisotropy class Σ_a); we then have $f(\alpha) + f(-\alpha) = 0$ for every $\alpha \in \Sigma$.

Let f' be the element of $Hom(X^*(T_0), \mathbb{Q})$ which coincides with f on $\pm\Sigma$; for every $\alpha \in \Phi$, we have $f'(\alpha) = \alpha(x)$ (remember that $\mathcal{A}_0 = X_*(T_0) \otimes \mathbb{R}$). We then have $f(\alpha) + f(-\alpha) = 0$ for every $\alpha \in \Phi$ if and only if f coincides with f' on Φ , and by definition of f , this is the case if and only if the image of f' is contained in $\frac{1}{2}\mathbb{Z}$. Proposition 6.1 is then an immediate consequence of the following proposition:

Proposition 6.2. *The function f being defined as above, the image of f' is actually contained in $\frac{1}{2}\mathbb{Z}$.*

Let β_1, \dots, β_d be the elements of Σ , and let α be any element of Φ , which we can assume to be different from the $\pm\beta_i$, $i \in \{1, \dots, d\}$. Write $\alpha = \sum_{i=1}^d \lambda_i \beta_i$, the λ_i being elements of \mathbb{Q} ; we then have $f'(\alpha) = \sum_{i=1}^d \lambda_i f'(\beta_i)$. On the other hand, for every i , we have:

$$\langle \alpha, \beta_i^\vee \rangle = \lambda_i \langle \beta_i, \beta_i^\vee \rangle = 2\lambda_i,$$

hence $\lambda_i \in \frac{1}{2}\mathbb{Z}$. Let (\cdot, \cdot) be a nontrivial W -invariant scalar product on $X^*(T) \otimes \mathbb{Q}$; we also have:

$$(1) \quad (\alpha, \alpha) = \sum_{i=1}^d \lambda_i^2 (\beta_i, \beta_i).$$

We now consider the possible cases. To simplify notations, we can assume that the nonzero λ_i are the ones with the lowest indices, and are positive (because we can always replace some of the β_i with their opposites by simply conjugating Σ by a product of reflections s_{β_i}).

- Assume first Φ is simply-laced. Then (α, α) and the (β_i, β_i) are all equal to each other, and there is only one possibility: $\lambda_i = \frac{1}{2}$ for $1 \leq i \leq 4$ and $\lambda_i = 0$ for $i > 4$; we then obtain:

$$f'(\alpha) = \frac{1}{2}(f'(\beta_1) + f'(\beta_2) + f'(\beta_3) + f'(\beta_4)) \in \frac{1}{2}(2 + \mathbb{Z}) = \frac{1}{2}\mathbb{Z}.$$

as desired.

- Assume now Φ is not simply-laced and every β_i such that $\lambda_i \neq 0$ is long; since there are then at least two long β_i orthogonal to each other, we cannot be in the case G_2 here. If α is long as well, we are reduced to the previous case. If α is short, then for every i , $(\alpha, \alpha) = \frac{1}{2}(\beta_i, \beta_i)$ and there is again only one possibility: $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $\lambda_i = 0$ for $i > 2$; we then obtain:

$$f'(\alpha) = \frac{1}{2}(f'(\beta_1) + f'(\beta_2)) \in \frac{1}{2}(1 + \mathbb{Z}) = \frac{1}{2}\mathbb{Z},$$

- Assume now Φ is not simply-laced and some of the β_i such that $\lambda_i \neq 0$ are short. We first treat the case G_2 ; in this case, assuming β_1 is short and β_2 is long, we have $(\beta_2, \beta_2) = 3(\beta_1, \beta_1)$, and $3\lambda_2^2 + \lambda_1^2$ is either 1 (if α is short) or 3 (if α is long). In the first (resp. second) case, it implies $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$ (resp. $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{3}{2}$), and in both cases, we obtain $f'(\alpha) = \lambda_1 f'(\beta_1) + \lambda_2 f'(\beta_2) \in \frac{1}{2}\mathbb{Z}$.
- Assume now Φ is not of type G_2 , not simply-laced and β_1 is short. First assume β_1 is the only short β_i such that $\lambda_i \neq 0$. If α is long, this is only possible if there are three nonzero λ_i , $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \frac{1}{2}$, and we then have:

$$f'(\alpha) = f'(\beta_1) + \frac{1}{2}(f'(\beta_2) + f'(\beta_3)) \in \frac{1}{2}\mathbb{Z} + \frac{1}{2}(1 + \mathbb{Z}) = \frac{1}{2}\mathbb{Z}.$$

Assume now α is short, still with only one of the β_i such that $\lambda_i \neq 0$ being short. We deduce from the relation (1) that we must have:

$$1 = \lambda_1^2 + 2 \sum_{i=2}^d \lambda_i^2.$$

Since λ_2 is nonzero, we must have $\lambda_1 = \frac{1}{2}$. But then the right-hand side of the above equality belongs to $\frac{1}{4} + \frac{1}{2}\mathbb{Z}$, hence cannot be equal to 1. We are then in an impossible case.

- Assume finally that at least two of the β_i such that the λ_i are nonzero are short. By lemma 5.9, this is possible only if Φ is of type C_d with $d \geq 4$.

On the other hand, by proposition 5.13, if Φ is of type C_d , Σ_a contains only long roots. Hence this case is impossible too.

□

Now we consider the cases where D is of positive dimension or in other words the ones where Σ_a contains less than d elements; as we have already seen, these cases are A_d , $d > 1$ odd (remember that we rule out the case A_{2n} in this whole section), D_d , $d = 2n + 1$ odd, and E_6 .

Proposition 6.3. *Assume D is of positive dimension. Then its vertices are all special.*

When G is of type A_d , we see on the table of page 29 of [7] that every vertex of X_E is special and the result of the proposition is trivial; we then only have to consider the cases D_{2n+1} and E_6 .

Assume Φ is of type D_{2n+1} . The facet D is then of dimension 1, and we have, for example, $\Sigma_a = \{\varepsilon_2 \pm \varepsilon_3, \dots, \varepsilon_{2n} \pm \varepsilon_{2n+1}\}$; Σ_a is then contained in the Levi subsystem Φ' of type D_{2n} of Φ generated by $\alpha_2, \dots, \alpha_{2n+1}$. Let Y be the subgroup of $X^*(T_0)$ generated by Σ_a , let f be a concave function from $\Phi \cap Y$ to $\frac{1}{2}\mathbb{Z}$ such that $f(\alpha) \in \mathbb{Z} + \frac{1}{2}$ for every $\alpha \in \pm\Sigma$, and let f' be the element of $\text{Hom}(Y, \mathbb{Q})$ which coincides with f on $\pm\Sigma$; if we extend f' linearly to $X^*(T_0) \otimes \mathbb{Q}$ by choosing $f'(\varepsilon_1)$ arbitrarily in $\frac{1}{2}\mathbb{Z}$, we obtain on Φ a concave function satisfying $f'(\alpha) + f'(-\alpha) = 0$ for every $\alpha \in \Phi$ and associated to some vertex of \mathcal{A}^Γ , and it is easy to check that every vertex of \mathcal{A}^Γ is associated to such a concave function, hence special.

Assume now Φ is of type E_6 . The facet D is then of dimension 2, and, up to conjugation, we have $\Sigma_a = \{\alpha_2, \alpha_3, \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$; Σ_a is then contained in the Levi subsystem Φ' of type D_4 of Φ generated by $\alpha_2, \dots, \alpha_5$. Once again, Y and f' being defined as in the previous case, we can extend f' linearly to $X^*(T_0) \otimes \mathbb{Q}$ by choosing $f'(\alpha_1)$ and $f'(\alpha_6)$ arbitrarily in $\frac{1}{2}\mathbb{Z}$, and we conclude similarly as above. □

Now assume the geometric realization of D is contained in \mathcal{A}_0 ; let $f_{D,E}$ be the concave function associated to D (as a facet of X_E ; we have to specify here since D may be a vertex of both X_E and X_F). The following corollary follows immediately from propositions 6.1 and 6.3:

Corollary 6.4. *Let α be an element of Φ which is a linear combination of elements of Σ . Then $f_{D,E}(\alpha) + f_{D,E}(-\alpha) = 0$.*

Let now A be any Γ -stable apartment of X_E of anisotropy class Σ_a containing D .

Proposition 6.5. *The subvcomplex A^Γ is isomorphic to any apartment of a building of type A_r , where r is its dimension.*

Let Φ_D be the root system of $K_{D,E}/K_{D,E}^0$, viewed as a root subsystem of Φ , and let $S_0 \subset T_0$ be the intersection of the $\text{Ker}(\alpha)$, $\alpha \in \Phi$. Assume there exists a field F_1 (not necessarily related to F in any way) on which G is defined and a

F_1 -inner form G' of G such that S_0 is precisely the maximal F_1 -split subtorus of T_0 in G' . Then by [2, corollary 5.8], the set of nonzero restrictions to S_0 of the elements of Φ is a root system, which implies that \mathcal{A}^Γ is isomorphic as an affine simplicial complex to an apartment of a building of the same type as that root system.

Now we check that the above assumption is true. If $r = 0$, then \mathcal{A}^F consists of a single vertex and the result is trivial; assume $r > 0$. We then obtain, with the help of [20, section 17]:

- when G is of type A_{2n-1} , we can take $F_1 = F$, and G' is then, up to isogeny, the group $GL_n(\mathcal{D})$, where \mathcal{D} is a quaternionic division algebra over F . The group G' is then of relative type A_{n-1} , hence \mathcal{A}^Γ is isomorphic to an apartment of type A_{n-1} ;
- when G is of type D_{2n+1} , G' is isogeneous to a special orthogonal group in $4n+2$ variables defined by a quadratic form of index 1, and F_1 is any field on which such a quadratic form exists (for example \mathbb{R} , in which case $G_{\mathbb{R}} = SO_{4n+1,1}(\mathbb{R})$, but not F this time); G' is then of relative type A_1 ;
- when G is of type E_6 , G' is the case (1, 6) of [20, proposition 17.7.2], and F_1 is any field on which such an inner form of G exists (again, $F_1 = F$ does not work, but $F_1 = \mathbb{R}$ does according to the classification of [21]); G' is then of relative type A_2 .

The proposition is now proved. \square

6.2. Restriction to Ch_D . Now we go back to the proof of theorem 1.2. Let A and A' be two Γ -stable apartments of X_E corresponding to tori of F -anisotropy class Σ_a . By proposition 4.7, there exists $g \in G_{F,der}$ such that $gA^\Gamma = A'^\Gamma$. If A^Γ (resp. A'^Γ) is a single vertex x (resp. x'), we have $Ch_{x',a} = gCh_{x,a}$, and the $G_{F,der}$ -invariance of the elements of $\mathcal{H}(X_E)^{G_{F,der}}$ implies that their restrictions to $Ch_{x,a}$ and $Ch_{x',a}$ are related. Assume now A^Γ and A'^Γ are of dimension at least 1. Then for every facet D of A^Γ of maximal dimension, gD is a facet of A'^Γ of maximal dimension, and the restriction to $Ch_{D,a}$ of every element of $\mathcal{H}(X_E)^{G_{F,der}}$ depends only on its restriction to $Ch_{gD,a}$ and conversely. To prove that $f \in \mathcal{H}(X_E)^{G_{F,der}}$ only depends on its restriction to $Ch_{D,a}$, we thus only need to prove that its restrictions to respectively $Ch_{D,a}$ and $Ch_{D',a}$, where D' is any other facet of maximal dimension of A^Γ , are related as well.

Proposition 6.6. *Let D, D' be two facets of maximal dimension of A^Γ . Then D and D' are $G_{F,der}$ -conjugates.*

The result is trivial when A^Γ consists of a single vertex; assume it is not the case. By an obvious induction we only have to prove the proposition when D and D' are neighboring each other. Let D'' be their common facet of codimension 1, and set $\mathbb{G}_D = K_{D,E}/K_{D,E}^1$; define similarly $\mathbb{G}_{D'}$ and $\mathbb{G}_{D''}$. It is easy to check (details are left to the reader) that:

- when G is of type A_{2n-1} , \mathbb{G}_D and $\mathbb{G}_{D'}$ are of type $A_1 \times A_1$ and $\mathbb{G}_{D''}$ of type A_3 ;
- when G is of type D_{2n+1} , \mathbb{G}_D and $\mathbb{G}_{D'}$ are of type D_{2n} and $\mathbb{G}_{D''}$ of type D_{2n+1} ;
- when G is of type E_6 , \mathbb{G}_D and $\mathbb{G}_{D'}$ are of type D_4 and $\mathbb{G}_{D''}$ of type D_5 .

Hence in every case (including the first one, remember that $D_3 = A_3$ and $D_2 = A_1 \times A_1$), \mathbb{G}_D and $\mathbb{G}_{D'}$ are of type D_{2r} and $\mathbb{G}_{D''}$ of type D_{2r+1} for some r ; we thus are reduced to the case D_{2n+1} , and we may assume G is SO'_{4n+2} . It is then easy to check that, depending on the case, D and D' are conjugated by some $G_{F,der}$ -conjugate of either:

$$\begin{pmatrix} & & & & & & 1 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \\ 1 & & & & & & & & \end{pmatrix},$$

or:

$$\begin{pmatrix} & & & & & & \varpi_F^{-1} \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \\ \varpi_F & & & & & & & & \end{pmatrix},$$

The proposition is then proved. \square

Corollary 6.7. *Let f be an element of $\mathcal{H}(X_E)^{G_{F,der}}$ and let D be the facet of X_E defined as in proposition 6.6. Then the restriction of f to Ch_a^0 is entirely determined by its restriction to $Ch_{D,a}$.*

Let D' be another facet of maximal dimension of some A'^Γ . If $g \in G_{F,der}$ is such that $A'^\Gamma = gA^\Gamma$, the restrictions of f to $Ch_{gD,a}$ and $Ch_{D',a}$ depend only on each other by the previous proposition and $G_{F,der}$ -invariance, and by $G_{F,der}$ -invariance again, its restrictions to $Ch_{D,a}$ and $Ch_{gD,a}$ are also linked. The result follows. \square

6.3. The harmonic cochains on $Ch_{D,a}$. Now we prove that, for some convenient D , the dimension of the space of the restrictions to $Ch_{D,a}$ of the elements of $\mathcal{H}(X_E)^{G_{F,der}}$ is at most 1.

We fix D arbitrarily for the moment among the possible ones contained in $A_{0,E}$. Let Φ_D be the root system of $K_{D,E}/K_{D,E}^0$ relative to $K_{T_0,E}/K_{T_0,E}^0$, viewed as a root subsystem of Φ .

Let β_1, \dots, β_r be the elements of some fixed representative of Σ_a , and let L be the subgroup of G generated by T_0 and the $U_{\pm\beta_i}$, $i = 1, \dots, r$; by proposition 4.5 we know that every E -split maximal F -torus of G of F -anisotropy class Σ_a is $G_{F,der}$ -conjugated to some maximal torus of L . Hence we can replace $Ch_{D,a}$ by the subset $Ch_{D,a,L}$ of the elements $C \in Ch_{D,a}$ contained in a Γ -stable apartment of X_E whose associated E -split maximal torus is also contained in L .

Proposition 6.8. *Let C be any chamber of A_E containing D ; there exist chambers C_0, C'_0 of $A_{0,E}$ containing D and corresponding to opposite Borel subgroups of $K_{D,E}/K_{D,E}^0$ and an element $u \in L_{E,der} \cap K_{C'_0,E}$ such that $C = uC_0$.*

Since T and T_0 are both contained in L , there exists $h \in L_{E,der}$ such that $hT_0h^{-1} = T$ and $hD = D$, hence $h \in K_D \cap L_{E,der}$. Therefore, we have $h^{-1}C = C_0$ for some chamber C_0 of $A_{0,E}$ containing D . Moreover, we have:

Lemma 6.9. *Let B, B' be two opposite Borel subgroups of $L_{E,der}$ containing T_0 and let U, U' be their respective unipotent radicals. Then T_0 and T' are conjugated by some element $h = uu'$ of $U_E U'_E$, where U_E (resp. U'_E) is the group of E -points of U (resp. U'). Moreover, if $h \in L_{E,der} \cap K_{D,E}$, then u and u' also belong to $L_{E,der} \cap K_{D,E}$.*

Let h' be any element of $L_{E,der}$ such that $h'T_0h'^{-1} = T'$. Using the Bruhat decomposition of L_E (see for example [20, 16.1.3]) and the fact that both B and B' contain T_0 , we can write $h' = unu''$, with $u, u'' \in U_E$ and $n \in N_{L_{E,der}}(T_0)$, and we can even assume that u'' belongs to $n^{-1}U'_E n$, hence $u' = nu''n^{-1} \in U'_E$; if we set $h = h'n^{-1}$, then $h = uu'$ satisfies $hT_0h^{-1} = h'T_0h'^{-1} = T'$, as required.

Assume now $h \in K_{D,E} \cap L_{E,der}$. Since the intersections of $K_{D,E}$ with respectively U_E and U'_E are products of the intersections with $K_{D,E}$ of the root subgroups respectively contained in these two subgroups, and since these two sets of root subgroups are disjoint, we obtain that u and u' belong to $K_{D,E} \cap L_{E,der}$ as well. \square

Note that since T_0 is split and T is of anisotropy class Σ_a , the element n of $N_{L_{E,der}}(T_0)$ used in the above proof always corresponds to the element of the Weyl group of L relative to T_0 which sends every root of L , hence also every root of $K_{D,E}$ by linearity, to its opposite (more precisely, w is the product of d copies of w_0 , where w_0 is the unique nontrivial element of the Weyl group of SL_2). Since h' has been chosen arbitrarily, we obtain that every $h \in L_{E,der}$ such that $hT_0h^{-1} = T'$ satisfies $h \in U_E U'_E T_{0,E}$, and that when h belongs to $K_{D,E}$, its three components also belong to $L_{E,der} \cap K_{D,E}$.

Now we prove proposition 6.8. According to lemma 6.9 and the previous remark, for every choice of U_E and U'_E , we have $C = uu'C_0$ for some C_0 , some $u \in L_{E,der} \cap U_E$ and some $u' \in L_{E,der} \cap U'_E$. Hence for every C_0 , if we choose U_E, U'_E in such a way that the image of u' in $K_{D,E}/K_{D,E}^0$ belongs to the Borel subgroup of $K_{D,E}/K_{D,E}^0$ corresponding to C_0 , or in other words that $u' \in K_{C_0,E}$, we have in fact $C = uC_0$. Let then C'_0 be the unique chamber of $A_{0,E}$ containing D and corresponding to a Borel subgroup of $K_{D,E}/K_{D,E}^0$ opposite to the previous one; by definition of U_E and by lemma 6.9, we must then have $u \in L_{E,der} \cap K_{C'_0,E}$, as required. \square

For every $\alpha \in \Phi$, let u_α be a group isomorphism between the additive group E and U_α compatible with the valued root datum $(G, T_0, (U_\alpha)_{\alpha \in \Phi}, (\phi_\alpha)_{\alpha \in \Phi})$; for every one-parameter subgroup ξ of T_0 , we then have, for every $x, y \in E^*$:

$$\xi(x)u_\alpha(y)\xi(x)^{-1} = u_\alpha(x^{<\alpha, \xi>}y),$$

where $<.,. >$ denotes the usual pairing between $X^*(T_0)$ and $X_*(T_0)$.

Corollary 6.10. *There exist elements $\lambda_1, \dots, \lambda_r \in \mathcal{O}_E^*$ such that the element u of lemma 6.9 is of the form $u = \prod_{i=1}^r u_{\beta_i}(\varpi_E^{2f_{D,E}(\beta_i)}\lambda_i)$ for some choice of $\Sigma_a = \{\beta_1, \dots, \beta_n\}$.*

(Note that since the elements of Σ_a are strongly orthogonal, the root subgroups U_{β_i} commute, hence the above product can be taken in any order.)

Assume Σ_a has been chosen in such a way that for every $\beta \in \Sigma_a$, the root subgroup U_β of G is contained in the group U_E defined as in proposition 6.8; since, using lemma 4.2, we can always replace some of its elements with their opposites, this is always possible.

Since u is unipotent, it belongs to the derived group $L_{E,der}$ of L_E , and we can work componentwise. Write $u = u_1 \dots u_r$, where for every i , u_i belongs to U_{β_i} . For every i , u_i then belongs to $K_{C'_0}$ but not to K_{C_0} , hence is of the form $u_{\beta_i}(\varpi_E^{2f_{D,E}(\beta_i)}\lambda_i)$ for some $\lambda_i \in \mathcal{O}_E^*$; the result follows. \square

Note that for every i , since $f_{D,E}(\beta_i) \in \mathbb{Z} + \frac{1}{2}$, $\varpi_E^{2f_{D,E}(\beta_i)}\lambda_i$ cannot be an element of F .

For every chamber C_0 of A_0 containing D , let Ch_{D,a,L,C_0} be the subset of the $C \in Ch_{D,a,L}$ such that, with C'_0 being defined as in proposition 6.8, $C = uC_0$ for some $u \in K_{C'_0,E}$. We deduce from the previous corollary that $Ch_{D,a,L}$ is the union of the Ch_{D,a,L,C_0} , with C_0 being such that the corresponding Borel subgroup of $K_{D,E}/K_{D,E}^0$ contains every root subgroup associated to the elements of some fixed representative of Σ_a .

Now we fix arbitrarily such a chamber C_0 . For every $\lambda_1, \dots, \lambda_r \in \mathcal{O}_E^*$, where r is the cardinality of Σ_a , let $C(\lambda_1, \dots, \lambda_r)$ be the chamber $\prod_{i=1}^r u_{\beta_i}(\varpi_E^{2f_{D,E}(\beta_i)}\lambda_i)C_0$, where the β_i are the elements of Σ_a . The chamber $C(\lambda_1, \dots, \lambda_r)$ only depends on the classes mod \mathfrak{p}_E of the λ_i , hence by a slight abuse of notation we can consider them as elements of the residual field $k_E^* = k_F^*$.

Proposition 6.11. *The subsets Ch_{D,a,L,C_0} of $Ch_{D,a,L}$ are all $K_{D,E} \cap G_{F,der}$ -conjugates.*

Let C_0, C'_0 be two chambers of $A_{0,E}$ containing D , and let C be any element of Ch_{D,a,L,C_0} ; there exists then $n \in N_{G_{E,der}}(T_0) \cap K_{D,E}$ such that $nC_0 = C'_0$. Let $g \in G_{E,der}$ be such that $gT_0g^{-1} = T$ and $gC_0 = C$, and set $n' = gng^{-1}$; the chamber $C' = n'C$ then belongs to Ch_{D,a,L,C'_0} . We thus only have to prove that C' is a $G_{F,der}$ -conjugate of some element of Ch_{D,a,L,C_0} .

By an obvious induction it is enough to prove the result when C and C' are neighboring each other. Assume first n is the reflection associated to some element β_i of Σ_a ; then C' is of the form $C' = C(\lambda_1, \dots, \lambda_{i-1}, \mu, \lambda_{i+1}, \dots, \lambda_r)$ for some $\mu \in k_F^*$ distinct from λ_i , and the result follows.

Next we prove the following lemma:

Lemma 6.12. *Let n' be any element of $N_{G_E}(T) \cap K_{D,E}$. Then $L' = nLn^{-1}$ is F -split.*

If we assume that L' is defined over F , then it is F -split by lemma 4.6. Therefore, we only have to prove that L' is defined over F . Let w be the element of the Weyl group of G/T corresponding to n : since T is of F -anisotropy class Σ_a , for every α belonging to the root system $\Phi_{L',T}$ of L' relative to T , we have $\gamma(\alpha) = -\alpha$, hence for every root β of L'/T , $w(\beta)$ is a root of L'/T and:

$$\gamma(w(\beta)) = -w(\beta) = w(-\beta)$$

is also a root of L'/T . Hence L' is Γ -stable, hence defined over F . \square

According to this lemma, replacing L by some $K_{D,F}$ -conjugate if needed, we see that the result of proposition 6.11 holds when n is the reflection associated to any conjugate of any element of Σ_a . Since by [4, §I, proposition 11], two roots of Φ of the same length are always conjugates, proposition 6.11 holds when Σ_a contains roots of every length.

Now assume n is any element of $K_{D,E} \cap N_{G_{E,der}}(T)$, and let g be an element of $G_{E,der}$ such that $gC_0 = C$. We then have:

$$C' = ngC_0 = g(g^{-1}ng)C_0 = gn_0C_0,$$

where $n_0 = g^{-1}ng$ is an element of $K_{D,E} \cap N_{G_{E,der}}(T_0)$, which we can assume to be in $G_{F,der}$ since T_0 is F -split. On the other hand, by lemma 6.9, g is of the form $(n_0un_0^{-1})(n_0u'n_0^{-1})$, with $u' \in K_{C_0}$ and u being of the form $u = \prod_{i=1}^r u_{\beta_i}(\varpi_E^{2f_{D,E}(\beta_i)} \mu_i)$, with μ_1, \dots, μ_r being elements of k_F^* . Hence $n_0^{-1}C' = uu'C_0$ belongs to $C(\mu_1, \dots, \mu_r)$ and C' is then $G_{F,der}$ -conjugated to some element of Ch_{D,a,L,C_0} .

It remains to prove that every n such that C and nC are neighboring each other is a G_F -conjugate of some element of $K_{D,E} \cap N_{G_{E,der}}(T)$. It is of course true when n is a representative of the reflection associated to some conjugate of some element of Σ_a , hence since Σ_a always contains some long roots, we only have to

consider the case where Σ_a contains only long roots and n is a representative of the reflection associated to some short root of Φ . Since Φ is then not simply-laced, we deduce from the remark following lemma 4.4 that Φ satisfies the equivalent conditions of that lemma, which implies in particular that Φ_D is of the same rank as Φ , and proposition 6.1 then implies that $\Phi_D = \Phi$. Hence n is a representative of the reflection associated to some element of Φ_D , and the result follows. \square

By the above proposition, to prove theorem 1.2, we only have to prove that the space of the restrictions of the elements of $\mathcal{H}(X_E)^{G_{F,der}}$ to Ch_{D,a,L,C_0} is of dimension at most 1 for some given D, a, L, C_0 . We start by dividing that set into $L_{F,der}$ -conjugacy classes, which happen to be easier to handle than the full $G_{F,der}$ -conjugacy classes.

Proposition 6.13. *The $L_{F,der}$ -conjugacy classes of elements of Ch_{D,a,L,C_0} are in 1-1 correspondence with the elements of the cohomology group $H^1(\Gamma, K_{T \cap L_{E,der}})$. Moreover, that group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where r is the cardinality of Σ_a .*

First we compute $H^1(\Gamma, K_{T \cap L_{E,der}})$. It is obvious from the definitions that the group $L_{E,der}$ is F -anisotropic, hence $T \cap L_{E,der}$ is nothing else than the F -anisotropic component of T . Let ξ be any 1-parameter subgroup of $T \cap L_{E,der}$; its intersection with $K_{T \cap L_{E,der}}$ is $\xi(\mathcal{O}_E^*)$. On the other hand, since $Im(\xi)$ is contained in T_a , for every $\lambda \in \mathcal{O}_E^*$, we have $\gamma(\xi(\lambda)) = \xi(\gamma(\lambda)^{-1})$. Hence $\xi(\lambda)$ defines a 1-cocycle of Γ if and only if $\gamma(\lambda)^{-1}\lambda = 1$, or in other words if and only if $\lambda \in \mathcal{O}_F^*$. (Note that it does not mean that $\xi(\lambda) \in G_F$.) Moreover, $\xi(\lambda)$ defines a 1-coboundary if and only if $\lambda = \gamma(\mu)\mu$ for some $\mu \in \mathcal{O}_E^*$, or in other words if and only if λ is the norm of some element of \mathcal{O}_E^* , which is true if and only if its image in k_F^* is a square. Since $X_*(T \cap L_{E,der})$ is generated by the coroots $\beta_1^\vee, \dots, \beta_r^\vee$ associated to the elements β_1, \dots, β_r of Σ_a , we obtain that $H^1(\Gamma, K_{T \cap L_{E,der}})$ is isomorphic to a product of r copies of $k_F^*/(k_F^*)^2 \simeq \mathbb{Z}/2\mathbb{Z}$.

Now we prove some lemmas.

Lemma 6.14. *Let F' be the unique quadratic unramified extension of F . Then the elements of Ch_{D,a,L,C_0} are all $L_{F',der}$ -conjugates.*

Let T be a maximal torus of G satisfying the conditions of lemma 6.15. By simply replacing F by F' in the discussion above, we obtain that when λ is an element of $k_{E'}^*$, $\xi(\lambda)$ defines a 1-cocycle in $K_{T \cap L_{E',der}}$ if and only if $\lambda \in k_{F'}^*$ and a 1-coboundary if and only if λ is the norm of an element of $k_{E'}^*$, which is true if and only if it is a square in $k_{F'}^*$. On the other hand, $[k_{F'}^* : k_F^*] = q + 1$ is even, hence every element of k_F^* is a square in $k_{F'}^*$. Lemma 6.14 follows. \square

Set $E' = EF'$; E'/E is then a quadratic unramified extension, and X_E is a simplicial subcomplex of the building $X_{E'}$ of $G_{E'}$; the set Ch_{D,a,L,C_0} is then a subset of the set of chambers of $X_{E'}$ containing D . Moreover, the extension E'/F' is quadratic and tamely ramified.

Lemma 6.15. *Assume C is an element of Ch_E ; let A be a Γ -stable apartment of X_E containing C , and let T be the corresponding E' -split torus of G . Then*

we can choose A in such a way that T is defined over F , E -split and that its F -anisotropic and F' -anisotropic components are identical.

Since $C \in Ch_E$, it is possible to choose A in such a way that A is contained in X_E , which, since it is Γ -stable, ensures that T is defined over F and E -split.

Moreover, since E'/F' is tamely ramified, the geometrical building $\mathcal{B}_{F'}$ of $G_{F'}$ is the set of Γ -fixed points of $\mathcal{B}_{E'}$, and in particular we have $\mathcal{B}_{F'} \cap \mathcal{B}_E = \mathcal{B}_F$. Hence the affine subspaces of $R(A)$ contained in respectively \mathcal{B}_F and $\mathcal{B}_{F'}$ are the same, which proves that the F -anisotropic and F' -anisotropic components of T have the same dimension. Since the second one is obviously contained in the first one, the lemma follows. \square

Now we prove the first assertion of proposition 6.13. For every $i \in \{1, \dots, r\}$, every $\lambda_1, \dots, \lambda_r \in \mathcal{O}_E^*$ and every $\mu \in \mathcal{O}_{E'}^*$ whose square is an element of \mathcal{O}_E^* , we have:

$$\beta_i^\vee(\mu)C(\lambda_1, \dots, \lambda_r) = C(\lambda_1, \dots, \mu^2\lambda_i, \dots, \lambda_r).$$

The chamber $C(\lambda_1, \dots, \lambda_r)$ being stable by $\beta_i^\vee(1 + \mathfrak{p}_{E'}) \subset K_{C(\lambda_1, \dots, \lambda_r), E'}$, we can assume $\mu \in \mathcal{O}_{F'}^*$, which implies $\mu^2 \in \mathcal{O}_E^* \cap \mathcal{O}_{F'}^* = \mathcal{O}_F^*$. Since every element of k_F^* is a square in $k_{F'}^*$, the image of μ^2 in k_F^* can be any element of k_F^* ; we thus obtain that the subgroup L of the elements of $(T_0)_F$ such that $tC(\lambda_1, \dots, \lambda_r)$ belongs to Ch_{D, a, L, C_0} , contains representatives of every element of $(k_F^*/(k_F^*)^2)^r \simeq H^1(\Gamma, K_{T \cap L_{E, der}})$; this proves that the set of $L_{F, der}$ -conjugacy classes of elements of Ch_{D, a, L, C_0} is in 1-1 correspondence with $H^1(\Gamma, K_{T \cap L_{F, der}})$, and proposition 6.13 is now proved. \square

For every $h = (\sigma_1, \dots, \sigma_r) \in H^1(\Gamma, K_{T, E} \cap L_{E, der})$, the σ_i being elements of $\mathbb{Z}/2\mathbb{Z}$ that we will denote by $+$ or $-$ signs in the sequel, let $Ch(h) = Ch(\sigma_1, \dots, \sigma_r)$ be the $L_{F, der}$ -conjugacy class of chambers of X_E containing the $C(\lambda_1, \dots, \lambda_r)$ such that for every i , λ_i is a square (resp. not a square) if $\sigma_i = +$ (resp. $\sigma_i = -$). Of course the $Ch(h)$ depend on the choices we have made for D and Σ_a .

We denote by (e_1, \dots, e_r) the canonical basis of $H^1(\Gamma, K_{T, E} \cap L_{E, der})$ viewed as a $\mathbb{Z}/2\mathbb{Z}$ -vector space. More precisely, for every $i \in \{1, \dots, r\}$, e_i is the element $(+, \dots, +, -, +, \dots, +)$, where the minus sign is in i -th position, and corresponds by the above correspondence to the root $\beta_i \in \Sigma_a$ (or in other words, $(\sigma_1, \dots, \sigma_r) \in H^1(\Gamma, K_{T, E} \cap L_{E, der})$ corresponds to elements of Ch_E of the form $u_{\beta_i}(\varpi_E^{2f_{D, E}(\beta_i)} \lambda_i)C_0$, where for every i , λ_i is a square if and only if $\sigma_i = +$).

By a slight abuse of notation, for every $h = (\sigma_1, \dots, \sigma_r) \in H^1(\Gamma, K_{T, E} \cap L_{E, der})$ and every $f \in \mathcal{H}(X_E)^{G_{F, der}}$, we write $f(h) = f(\sigma_1, \dots, \sigma_r)$ for the constant value of f on $Ch(\sigma_1, \dots, \sigma_r)$.

In the whole beginning of this section, D and C_0 have been chosen arbitrarily among the ones satisfying the required conditions. (We did not impose explicitly any particular conditions on Σ_a either, but we of course still assume Σ_a is the one given by either proposition 5.13 or proposition 5.14 depending on the case.) Now it is time to make more precise choices. Let then D be such that Φ_D is a standard Levi subsystem of Φ ; every element of Φ_D is then a sum of simple roots

contained in Φ_D . Let C_0 be the chamber of $A_{0,E}$ corresponding to the following concave function: for every $\alpha \in \Phi^+$, define $h(\alpha)$ the following way:

- if Σ_a contains roots of every length, then $h(\alpha)$ is the number of simple roots (counted with multiplicities) α is the sum of;
- if Φ is not simply-laced and Σ_a contains only long roots, $h(\alpha)$ is the number of long roots (again, counted with multiplicities) among the simple roots α is the sum of.

Note that we see from proposition 5.13 that the case where Φ is not simply-laced and Σ_a contains only short roots cannot happen.

Set $f(\alpha) = -\frac{h(\alpha)}{2}$. Set also $f(-\alpha) = \frac{h(\alpha)+1}{2}$. It is easy to check that f is concave; details are left to the reader. Moreover, since f is concave and $f(\alpha) + f(-\alpha) = \frac{1}{2}$ for every α , f is the concave function f_{C_0} associated to some chamber C_0 of $A_{0,E}$. We can also easily check that the extended set of simple roots associated to C_0 is $\Delta \cup \{-\alpha_0\}$. (Note that $R(C_0)$ is not contained in $R(C_{0,F})$ in general.)

For every $\alpha < 0$ which is the inverse of the sum of an odd number of simple roots in Φ^+ , $f_{C_0}(\alpha)$ is an integer, hence when $\Sigma_a = \{\beta_1, \dots, \beta_r\}$ contains roots of every length, we see with the help of proposition 5.15 that $f_{C_0}(\beta_i)$ is an integer for every i . Now we check that it is also true when Φ is not simply-laced and Σ contains only long roots. In that case, the assertion is an immediate consequence of proposition 5.15 and the following lemma:

Lemma 6.16. *Assume Φ is of type B_d , C_d or F_4 . Let α be a positive long root, and write $\alpha = \sum_{i=1}^d \lambda_i \alpha_i$, with $\alpha_1, \dots, \alpha_d$ being the elements of Δ . Then for every i such that α_i is short, λ_i is even.*

We prove the result by induction on $h(\alpha)$. If $h(\alpha) = 1$, then α is a long simple root and the result is trivial. Assume $h(\alpha) > 1$ and let i be such that $\alpha - \alpha_i$ is a root. If α_i is long, then $\alpha - \alpha_i$ is also long and positive and $h(\alpha - \alpha_i) = h(\alpha) - 1$; the result then follows from the induction hypothesis. Assume now α_i is short. Then α and α_i generate a subsystem of type B_2 of Φ , which implies in particular, since α_i is a simple root and $\alpha \neq \alpha_i$, that $\alpha - 2\alpha_i$ is also a positive root and is long. The result then follows from the induction hypothesis applied to $\alpha - 2\alpha_i$. \square

Now we prove that for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$, the $f(h)$, $h \in H^1(\Gamma, K_{T,E} \cap L_{E,der})$, are all determined by $f(1)$. We then establish relations between the $f(h)$ using the $G_{F,der}$ -invariance of f and the following two lemmas:

Lemma 6.17. *Let i be an element of $\{1, \dots, r\}$; assume β_i is the negative of a simple root in Φ^+ . Then for every $h \in H^1(\Gamma, K_{T,E} \cap L_{E,der})$ and every $f \in \mathcal{H}(X_E)^{G_{F,der}}$, $f(e_i h) = -f(h)$.*

Let $C = C(\lambda_1, \dots, \lambda_r)$ be an element of $Ch(h)$. Set $\mathbb{G}_D = K_{D,E}/K_{D,E}^0$ and let \mathbb{P}_i be the parabolic subgroup of \mathbb{G}_D generated by \mathbb{B}_0 and the root subgroup U_{β_i} .

Let $K_i \subset K_D$ be the corresponding parahoric subgroup of G_E and let D'_i be the codimension 1 facet of X_E associated to K_i . The chambers of X_E admitting D'_i as a wall are precisely the ones corresponding to the Iwahori subgroups contained in K_i . Out of these $q + 1$ chambers, two do not belong to $Ch_{D,a,L}$ (the chamber $C = "C(\lambda_1, \dots, \lambda_{i-1}, 0, \lambda_{i+1}, \dots, \lambda_r)"$ (with a slight abuse of notation) and $C' = s_{\beta_i}(C)$), which implies that every element of $\mathcal{H}(X_E)^{G_F, der}$ is zero on them, and the remaining $q - 1$ are the ones of the form $C(\lambda_1, \dots, \lambda_{i-1}, \mu, \lambda_{i+1}, \dots, \lambda_r)$ with $\mu \in k_F^*$; since exactly half of the elements of k_F^* are squares, the lemma follows immediately from the harmonicity condition. \square

Lemma 6.18. *Let β_i, β_j be two elements of Σ_a satisfying the following conditions:*

- $\alpha = \frac{\beta_i - \beta_j}{2}$ is a root, and β_j and α generate a subsystem of Φ of type B_2 ;
- α is the negative of a simple root of Φ^+ , and $f_{D,E}(\alpha)$ is an integer.

Then for every $h \in H^1(\Gamma, K_{T,E} \cap L_{E,der})$ and every $f \in \mathcal{H}(X_E)^{G_F, der}$, $f(e_i e_j h) = -f(h)$.

We first remark that by corollary 6.4, we have $f_{D,E}(\alpha) + f_{D,E}(-\alpha) = 0$, hence if $f_{D,E}(\alpha)$ is an integer, $f_{D,E}(-\alpha)$ is an integer as well.

Set $\mathbb{G}_D = K_{D,E}/K_{D,E}^0$, let \mathbb{T}_0 be the image of K_{T_0} in \mathbb{G}_D and let \mathbb{B}_0 be the Borel subgroup of \mathbb{G}_D containing \mathbb{T}_0 associated to Φ^+ . The root $-\alpha$, being a simple root in Φ^+ , is also a simple root in \mathbb{G}_D in the set of positive roots associated to \mathbb{B}_0 . Let \mathbb{P}' be the parabolic subgroup of \mathbb{G}_D generated by \mathbb{B}_0 and the root subgroup associated with $-\alpha$, and let K and D' be defined as the K_i and D_i of lemma 6.17, relatively to \mathbb{P}' this time. The chambers of X_E admitting D' as a wall are the ones of the form:

$$C_l = \left(\prod_{i=1}^r u_{-\beta_i}(\lambda_i) \right) l C_0,$$

where l is an element of the Levi component \mathbb{M}' of \mathbb{P}' , which is the product of a subgroup \mathbb{M}'' of type A_1 by the image of K_{T_0} in \mathbb{G}_D ; since K_{T_0} stabilizes C_0 we can assume that $l \in \mathbb{M}''$, and to simplify notations we can consider l as an element of $GL_2(k_F)$. On the other hand, l admits representatives in G_F , hence:

$$f(C_l) = f(l^{-1}C_l) = f(l^{-1} \left(\prod_{i=1}^r u_{-\beta_i}(\lambda_i) \right) l C_0).$$

Since conjugating $\prod_{i=1}^r u_{-\beta_i}(\lambda_i)$ by l leaves every term of the product but the i -th and j -th unchanged, we are reduced to the case where $d = 2$ and Φ itself is of type B_2 , in which case G is the group $SO_5 = PGSp_4$.

It turns out to be more convenient to work with $G = GSp_4$. The harmonicity condition applied to the chambers containing D' can then, up to G_F -conjugation

of the involved chambers, be rewritten as follows, if $h = (\sigma_1, \sigma_2)$:

$$(2) \quad \sum_{l \in R} f \left(\left(\begin{pmatrix} Id & 0 \\ \tau_l \begin{pmatrix} 0 & \sigma_1 \varpi_E \\ \sigma_2 \varpi_E & 0 \end{pmatrix} l & Id \end{pmatrix} C_0 \right) \right) = 0,$$

where R is a set of representatives of the right classes of $GL_2(k_F)$ modulo \mathbb{B}_0 . We thus have to find a set R such that for every $l \in R$, if C'_l is the chamber defined in the above sum, either C'_l belongs to $Ch(h')$ for some $h' \in H^1(\Gamma, K_{T,E} \cap L_{E,der})$ or $f(C'_l) = 0$.

To simplify the notations, we only write down the proof of the case $h = 1$; the other cases can be treated in a similar way. If $l = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q)$, then we have:

$$\tau_l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} l = \begin{pmatrix} ab + cd & a^2 + c^2 \\ b^2 + d^2 & ab + cd \end{pmatrix},$$

which means that we only have to consider the C_l such that there exists $l' \in \mathbb{B}$ such that ll' satisfies the condition $ab + cd = 0$; since that condition is obviously right \mathbb{T} -invariant we can even assume that l' is unipotent. A simple computation shows that in this case, $a^2 + c^2$ and $b^2 + d^2$ are either both squares or both non-squares, which implies that C_l belongs to either $Ch(1)$ or $Ch(e_1 e_2)$.

Consider first the element $l_\infty = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This element satisfies $a^2 + c^2 = b^2 + d^2 = 1$, hence we have $C_{l_\infty} \in Ch_x(1)$. Moreover, none of the $l_\infty u$, with $u \neq 0$ belonging to the unipotent radical \mathbb{U} of \mathbb{B}_0 , satisfies the condition $ab + cd = 0$.

Consider now, for every $y \in k_F$, the element $l_y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ of $GL_2(k_F)$. Another simple computation shows that there exists an element of $l_y \mathbb{U}$ satisfying the condition $ab + cd = 0$ if and only if $1 + y^2 \neq 0$, and that in that case, $\begin{pmatrix} 1 & \frac{-y}{1+y^2} \\ y & \frac{1}{1+y^2} \end{pmatrix}$ is the only such element. To prove the lemma, we now only have to compute the number of $y \in k$ such that $1 + y^2 = a^2 + c^2$ is nonzero and a square (resp. not a square).

Assume there exists $e \in k_F^*$ such that $1 + y^2 = e^2$; we then have $(e+y)(e-y) = 1$. Set $\lambda = e + y$; we then have $\lambda(\lambda - 2y) = 1$, hence $\lambda - \frac{1}{\lambda} = 2y$. Moreover, it is easy to check that $\lambda - \frac{1}{\lambda} = \mu - \frac{1}{\mu}$ if and only if either $\lambda = \mu$ or $\lambda = -\frac{1}{\mu}$.

Assume first -1 is not a square in k_F^* . Then $1 + y^2$ is always nonzero, and we only have to count the number of different y such that $1 + y^2$ is a square. On the other hand, we never have $\lambda = -\frac{1}{\lambda}$, hence for every $y \in k_F$, there are always either 0 or 2 values of λ such that $\lambda - \frac{1}{\lambda} = 2y$. Hence the number of possible values for y is $\frac{q-1}{2}$, which proves that for a suitable choice of R , taking into account l_∞ , there are exactly $\frac{q+1}{2}$ terms in the sum such that $C_l \in Ch(1)$ (resp. $C_l \in Ch(e_1 e_2)$). The lemma then follows immediately from the harmonicity condition.

Assume now -1 is a square in k_F^* . Then each one of its square roots λ satisfies $\lambda = -\frac{1}{\lambda}$ and is its own image by $\lambda \mapsto \frac{1}{2}(\lambda - \frac{1}{\lambda})$, hence by the previous remark is also its only inverse image by that same map. On the other hand, every y such that $y^2 \neq -1$ has either 0 or 2 inverse images, hence there are exactly $\frac{q+1}{2}$ elements y such that $y^2 + 1$ is a square, $\frac{q-3}{2}$ of them not being roots of -1 , and $\frac{q-1}{2}$ elements y such that $y^2 + 1$ is not a square. Taking into account l_∞ once again, we conclude as in the previous case. \square

Now we use these lemmas to prove theorem 1.2. We already know that every $f \in \mathcal{H}(X_E)^{G_{F,der}}$ is entirely determined by the $f(h)$, $h \in H^1(\Gamma, K_{T \cap L_{E,der}})$; it then only remains to prove the following proposition:

Proposition 6.19. *Let f be any element of $\mathcal{H}(X_E)^{G_{F,der}}$, viewed as a function on $H^1(\Gamma, K_{T \cap L_{E,der}})$. Then f is entirely determined by $f(1)$.*

If $\lambda_1, \dots, \lambda_r$ are elements of k^* such that $C(\lambda_1, \dots, \lambda_r) \in G_{FC}(1, \dots, 1)$, and if h' is the element of $H^1(\Gamma, K_{T \cap L_{E,der}})$ corresponding to the elements $\lambda_1, \dots, \lambda_r$, then we have $f(h'h) = f(h)$ for every $h \in H^1(\Gamma, K_{T \cap L_{E,der}})$. Moreover, if i is such that β_i is the negative of a simple root, by lemma 6.17, setting $h' = e_i$, $f(h'h) = -f(h)$ for every $h \in H^1(\Gamma, K_{T \cap L_{E,der}})$. Finally, if β_i, β_j are two elements of Σ_a satisfying the conditions of lemma 6.18, then by that lemma, setting $h' = e_i e_j$, we have $f(h'h) = -f(h)$ for every $h \in H^1(\Gamma, K_{T \cap L_{E,der}})$. We thus only have to prove that the set S of all these various elements h' always generates $H^1(\Gamma, K_{T \cap L_{E,der}})$ as a $\mathbb{Z}/2\mathbb{Z}$ -vector space.

We proceed by a case-by-case analysis. In the rest of the proof, the α_i and the ε_i are defined the same way as in [4, plates I to IX].

- Assume first Φ is of type A_d , with $d = 2n - 1$ being odd; by proposition 5.14, Φ_D is then the Levi subsystem of Φ generated by the simple roots α_{2i-1} , $i = 1, \dots, n$, and we can set for every i $\beta_i = -\alpha_{2i-1}$, which is always the negative of a simple root of Φ^+ ; by lemma 6.17, for every $i \in \{1, \dots, n\}$, $e_i \in S$ and $f(e_i) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. The result follows.
- Assume now Φ is of type B_d , with $d = 2n$ being even; we have $\Phi_D = \Phi$. By proposition 5.13, for every $i \in \{1, \dots, n\}$, we can set $\beta_{2i-1} = -\varepsilon_{2i-1} - \varepsilon_{2i}$ and $\beta_{2i} = -\varepsilon_{2i-1} + \varepsilon_{2i}$. The β_{2i} are then negatives of simple roots of Φ^+ , hence by lemma 6.17, for every i , $e_{2i} \in S$ and $f(e_{2i}) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. Moreover, for every element of Φ of the form $\alpha = \varepsilon_{2i} + \varepsilon_{2i+1}$, it is easy to check that $\langle \beta_j, \alpha^\vee \rangle$ is odd if and only if $j \in \{2i-1, 2i, 2i+1, 2i+2\}$, hence if c is an element of \mathcal{O}_F^* which is not a square, $\alpha^\vee(c)$ acts on $H^1(\Gamma, K_{T \cap L_{E,der}})$ by multiplication by $e_{2i+1}e_{2i}e_{2i+1}e_{2i+2}$, which implies that $e_{2i-1}e_{2i}e_{2i+1}e_{2i+2} \in S$ and $f(e_{2i-1}e_{2i}e_{2i+1}e_{2i+2}) = f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. We thus have obtained $2n - 1$ linearly independent elements of S ; we still need one more.

We will now prove that $e_{2n+1}e_{2n} \in S$ and $f(e_{2n+1}e_{2n}) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. Let $\alpha = \varepsilon_n$ be the unique short simple root in Φ^+ ; the roots β_{2n-1} , β_{2n} and α then satisfy the conditions of lemma 6.18, and the desired result follows.

- Assume now Φ is of type B_d , with $d = 2n + 1$ being odd; we have $\Phi_D = \Phi$. By proposition 5.13, we can define the β_i , $i \leq 2n$, as in the previous case and set $\beta_d = -\varepsilon_d$. Then for j being either an even integer or d , β_j is the negative of a simple root, hence by lemma 6.17 $e_j \in S$ and $f(e_j) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$; moreover, for every $i \in \{1, \dots, n-1\}$, we obtain $e_{2i-1}e_{2i}e_{2i+1}e_{2i+2} \in S$ and $f(e_{2i-1}e_{2i}e_{2i+1}e_{2i+2}) = f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$ by the same reasoning as in the previous case; we also similarly obtain $e_{d-2}e_{d-1}e_d \in S$ and $f(e_{d-2}e_{d-1}e_d) = f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. This makes $2n + 1$ linearly independent elements of S , as desired.
- Assume now Φ is of type C_d ; we have $\Phi_D = \Phi$. By proposition 5.13, we can set $\beta_i = -2\varepsilon_i$ for every i . The root β_d is the negative of a simple root, hence $e_d \in S$ and $f(e_d) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$; moreover, for every $i \in \{1, \dots, d-1\}$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ is a simple root and β_i , β_{i+1} and α_i satisfy the conditions of lemma 6.18, hence $e_i e_{i+1} \in S$ and $f(e_i e_{i+1}) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. We thus obtain d linearly independent elements of S , as desired.
- Assume now Φ is of type D_d , with $d = 2n$ being even; we have $\Phi_D = \Phi$. By proposition 5.13, we can choose the β_i the same way as in the case B_{2n} , and it is easy to check that the first $2n - 1$ linearly independent elements of S are the same, with the same relative values of $f \in \mathcal{H}(X_E)^{G_{F,der}}$; to get one more, we simply remark that $-\beta_{2n-1}$ is now also the negative of a simple root of Φ^+ , which implies that $e_{2n-1} \in S$ and $f(e_{2n-1}) = -f(1)$.
- Assume now Φ is of type D_d , with $d = 2n + 1$ being odd; we deduce from proposition 5.14 that Φ_D is then the Levi subsystem of Φ generated by the simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 2, \dots, d-1$, and $\alpha_d = \varepsilon_{d-1} + \varepsilon_d$, and Σ_a is of cardinality $2n$. By that same proposition, the β_i are defined the same way as in the cases B_{2n} and D_{2n} , except that we add 1 to every index of the ε_i (i.e. ε_i becomes ε_{i+1}): more precisely, we now have $\beta_{2i-1} = -\varepsilon_{2i} - \varepsilon_{2i+1}$ and $\beta_{2i} = -\varepsilon_{2i} + \varepsilon_{2i+1}$ for every $i \in \{1, \dots, n\}$. The $2n$ linearly independent elements of S and the relative values of $f \in \mathcal{H}(X_E)^{G_{F,der}}$ are obtained as in the case D_{2n} , taking into account the shift of indices.
- Assume now Φ is of type E_6 ; by proposition 5.14, Φ_D is then the Levi subsystem of Φ generated by $\alpha_2, \dots, \alpha_5$, and Σ_a is of cardinality 4. By that same proposition, we can set $\beta_1 = -\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5$, $\beta_2 = -\alpha_2$, $\beta_3 = -\alpha_3$ and $\beta_4 = -\alpha_5$. Then β_2 , β_3 and β_4 are negatives of simple roots, hence for every $i \in \{2, 3, 4\}$, $e_i \in S$ and $f(e_i) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. Moreover, it is easy to check that $\langle \beta_i, \alpha_4^\vee \rangle$ is odd for

every i , hence if c is an element of \mathcal{O}_E^* , which is not a square, we have the following coroot action on $H^1(\Gamma, K_{T \cap L_{E, der}})$:

$$\alpha_4^\vee(c)h = e_1 e_2 e_3 e_4 h.$$

Hence $e_1 e_2 e_3 e_4 \in S$ and $f(e_1 e_2 e_3 e_4) = f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F, der}}$. This makes 4 linearly independent elements of S , as desired.

- Assume now Φ is of type E_7 ; we have $\Phi_D = \Phi$. By proposition 5.13, we can set $\beta_1 = -\alpha_0$, $\beta_2 = -\alpha_2$, $\beta_3 = -\alpha_3$, $\beta_4 = -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7$, $\beta_5 = -\alpha_5$, $\beta_6 = -\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5$ and $\beta_7 = -\alpha_7$. For every $i \in \{2, 3, 5, 7\}$, β_i is the negative of a simple root, hence by lemma 6.17 $e_i \in S$ and $f(e_i) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F, der}}$; on the other hand, we have:

- $\langle \beta_i, \alpha_1^\vee \rangle$ is odd if and only if $i = 1, 3, 4, 6$;
- $\langle \beta_i, \alpha_4^\vee \rangle$ is odd if and only if $i = 2, 3, 5, 6$;
- $\langle \beta_i, \alpha_6^\vee \rangle$ is odd if and only if $i = 4, 5, 7$;

hence if c is an element of \mathcal{O}_E^* which is not a square, we have the following coroot actions on $H^1(\Gamma, K_{T \cap L_{E, der}})$:

- $\alpha_1^\vee(c)h = e_1 e_3 e_4 e_6 h$;
- $\alpha_4^\vee(c)h = e_2 e_3 e_5 e_6 h$;
- $\alpha_6^\vee(c)h = e_4 e_5 e_7 h$,

Hence $e_1 e_3 e_4 e_6$, $e_2 e_3 e_5 e_6$ and $e_4 e_5 e_7$ belong to S ; for every $f \in \mathcal{H}(X_E)^{G_{F, der}}$, the value of f on them is then equal to $f(1)$. We thus obtain 7 linearly independent elements of S , as desired.

- Assume now Φ is of type E_8 ; we have $\Phi_D = \Phi$. By proposition 5.13, we can set $\beta_1 = -\alpha_0$, $\beta_2 = -\alpha_2$, $\beta_3 = -\alpha_3$, $\beta_4 = -2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7$, $\beta_5 = -\alpha_5$, $\beta_6 = -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7$, $\beta_7 = -\alpha_7$ and $\beta_8 = -\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5$. For every $i \in \{2, 3, 5, 7\}$, as in the case E_7 , β_i is the negative of a simple root, hence by lemma 6.17 $e_i \in S$ and $f(e_i) = -f(1)$ for every i ; on the other hand, we have:

- $\langle \beta_i, \alpha_1^\vee \rangle$ is odd if and only if $i = 3, 4, 6, 8$;
- $\langle \beta_i, \alpha_4^\vee \rangle$ is odd if and only if $i = 2, 3, 5, 8$;
- $\langle \beta_i, \alpha_6^\vee \rangle$ is odd if and only if $i = 5, 6, 7$;
- $\langle \beta_i, \alpha_8^\vee \rangle$ is odd if and only if $i = 1, 4, 6, 7$;

hence if c is an element of \mathcal{O}_E^* which is not a square, we have the following coroot actions on $H^1(\Gamma, K_{T \cap L_{E, der}})$:

- $\alpha_1^\vee(c)h = e_3 e_4 e_6 e_8 h$;
- $\alpha_4^\vee(c)h = e_2 e_3 e_5 e_8 h$;
- $\alpha_6^\vee(c)h = e_5 e_6 e_7 h$;
- $\alpha_8^\vee(c)h = e_1 e_4 e_6 e_7$.

Hence $e_2 e_3 e_4 e_6$, $e_4 e_5 e_6 e_7$, $e_3 e_7 e_8$ and $e_1 e_2 e_3 e_8$ belong to S and for every $f \in \mathcal{H}(X_E)^{G_{F, der}}$, the value of f on them is equal to $f(1)$. We thus obtain 8 linearly independent elements of S , as desired.

- Assume now Φ is of type F_4 ; we have $\Phi_D = \Phi$. By proposition 5.13, we can set $\beta_1 = -\alpha_0$, $\beta_2 = -\alpha_2$, $\beta_3 = -\alpha_2 - 2\alpha_3$ and $\beta_4 = -\alpha_2 - 2\alpha_3 - 2\alpha_4$. Since $(\alpha_2, \alpha_3, \alpha_4)$ is the set of simple roots of a standard Levi subsystem of type C_3 of Φ , with the help of the case C_d applied to that subsystem, we obtain that e_2, e_3 and e_4 belong to S and $f(e_4) = -f(e_3) = f(e_2) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$; on the other hand, $\langle \beta_i, \alpha_1^\vee \rangle$ is odd for every i , hence if c is an element of \mathcal{O}_E^* which is not a square, we have the following coroot action on $H^1(\Gamma, K_{T \cap L_{E,der}})$:

$$\alpha_1^\vee(c)h = e_1 e_2 e_3 e_4 h.$$

Hence $e_1 e_2 e_3 e_4$ belongs to S as well, and $f(e_1 e_2 e_3 e_4) = f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. The result follows.

- Assume finally Φ is of type G_2 ; we have $\Phi_D = \Phi$. By proposition 5.13, we can set $\beta_1 = -\alpha_1$ and $\beta_2 = -\alpha_0$; β_1 is then the negative of a simple root, hence by lemma 6.17 $e_1 \in S$ and $f(e_1) = -f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$; on the other hand, $\langle -\alpha_0, \alpha_2^\vee \rangle$ and $\langle -\alpha_1, \alpha_2^\vee \rangle$ are both odd, hence if c is an element of \mathcal{O}_E^* which is not a square, we have the following coroot action on $H^1(\Gamma, K_{T \cap L_{E,der}})$:

$$\alpha_2^\vee(c)h = e_1 e_2 h.$$

Hence $e_1 e_2$ belongs to S and $f(e_1 e_2) = f(1)$ for every $f \in \mathcal{H}(X_E)^{G_{F,der}}$. The result follows.

The proposition is now proved. \square

Corollary 6.20. *Assume Φ is not of type A_{2n} for any n . Then theorem 1.2 holds.*

6.4. Action of some elements of G_F . We finish this section by summarizing the action of the simple coroots of Φ^+ on $H^1(\Gamma, K_{T \cap L_{E,der}})$ (proposition 6.21) and the elements of the canonical basis of $H^1(\Gamma, K_{T \cap L_{E,der}})$ on $\mathcal{H}(X_E)^{G_{F,der}}$ (proposition 6.22):

Proposition 6.21. *Assume Φ is not of type A_{2n} for any n ; let h be an element of $H^1(\Gamma, K_{T \cap L_{E,der}})$, and let c be an element of k_F^* which is not a square. We have:*

- if Φ is of type A_{2n-1} :
 - $\alpha_{2i+1}^\vee(c)h = h$ for every i ;
 - $\alpha_{2i}^\vee(c)h = e_i e_{i+1} h$ for every i .
- if Φ is of type B_d :
 - $\alpha_i^\vee(c)h = h$ if either i is odd or $i = d$;
 - $\alpha_i^\vee(c)h = e_{i-1} e_i e_{i+1} e_{i+2} h$ if i is even and $i < d-1$;
 - $\alpha_{d-1}^\vee(c)h = e_{d-2} e_{d-1} e_d$ if d is odd.
- if Φ is of type C_n , $\alpha_i^\vee(c)h = h$ for every i ;
- if Φ is of type D_n :

- $\alpha_{d-i}^\vee(c)h = h$ for every odd i ;
- $\alpha_{d-i}^\vee(c)h = e_{d-i-1}e_{d-i}e_{d-i+1}e_{d-i+2}h$ for every i even, positive and such that $d-i > 1$;
- $\alpha_d^\vee(h) = h$;
- when d is odd, $\alpha_1^\vee(c)h = e_1e_2h$.
- if Φ is of type E_6 :
 - $\alpha_i^\vee(c)h = h$ for every $i < 4$;
 - $\alpha_4^\vee(c)h = e_1e_2e_3e_4h$;
- if Φ is of type E_7 :
 - $\alpha_i^\vee(c)h = h$ for $i = 2, 3, 5, 7$;
 - $\alpha_1^\vee(c)h = e_1e_3e_4e_6h$;
 - $\alpha_4^\vee(c)h = e_2e_3e_5e_6h$;
 - $\alpha_6^\vee(c)h = e_4e_5e_7h$,
- if Φ is of type E_8 :
 - $\alpha_i^\vee(c)h = h$ for $i = 2, 3, 5, 7$;
 - $\alpha_1^\vee(c)h = e_3e_4e_6e_8h$;
 - $\alpha_4^\vee(c)h = e_2e_3e_5e_8h$;
 - $\alpha_6^\vee(c)h = e_5e_6e_7h$;
 - $\alpha_8^\vee(c)h = e_1e_4e_6e_7$.
- if Φ is of type F_4 :
 - $\alpha_1^\vee(c)h = e_1e_2e_3e_4h$;
 - $\alpha_i^\vee(c)h = h$ for every $i \geq 2$;
- if Φ is of type G_2 , $\alpha_1^\vee(c)h = h$ and $\alpha_2^\vee(c)h = e_1e_2h$.

Proposition 6.22. *Assume Φ is not of type A_{2n} for any n . Let f be a nonzero element of $\mathcal{H}(X_E)^{G_{F, \text{der}}}$; we have:*

- if Φ is of type A_{2n-1} , $f(e_i) = -f(1)$ for every i ;
- if Φ is of type B_d , $f(e_i) = -f(1)$ if either $i = d$ or i is even, and $f(e_i) = f(1)$ if i is odd and $< d$;
- if Φ is of type C_d , $f(e_i) = (-1)^{d+1-i}f(1)$ for every i ;
- if Φ is of type D_d (either odd or even), $f(e_i) = -f(1)$ for every i ;
- if Φ is of type E_6 , $f(e_i) = -f(1)$ for every i ;
- if Φ is of type E_7 , $f(e_i) = f(1)$ if i is either 1 or 2 and $-f(1)$ if $i \geq 3$;
- if Φ is of type E_8 , $f(e_i) = f(1)$ if i is either 1 or 3 and $-f(1)$ in the other cases;
- if Φ is of type F_4 , $f(e_i) = -f(1)$ for every i ;
- if Φ is of type G_2 , $f(e_i) = -f(1)$ for every i .

These relations either are already contained in the proof of proposition 6.19 or can be deduced from the relations established during that proof by easy computations. Details are left to the reader. \square

7. PROOF OF THE χ -DISTINCTION

7.1. A convergence result. Now we go to the proof of theorem 1.1. Before defining our linear form λ , we have to prove a preliminary result, which plays here the same role as [5, lemma 4.5] for the unramified case, except that it now works for any value of q thanks to the use of the Poincaré series.

To make notations clearer, we denote by $d_E(.,.)$ (resp. $d_F(.,.)$) the combinatorial distance between two chambers of X_E (resp. X_F).

Proposition 7.1. *Let f be an element of $\mathcal{H}(X_E)^\infty$, and let O be any G_F -orbit of chambers of X_E . Then we have:*

$$\sum_{C \in O} |f(C)| < +\infty.$$

Fix an element C of O . Let C_0 be an element of Ch_E whose geometric realization is contained in \mathcal{B}_F and such that $d_E(C, C_0)$ is minimal, and let C_F be the chamber of X_F whose geometric realization contains $R(C_0)$. We first prove the following lemmas:

Lemma 7.2. *Let A_F be an apartment of X_F containing C_F and let T be the associated F -split torus of G . For every $t \in T_F$, we have $d_E(C_0, tC_0) = 2d_F(C_F, tC_F)$.*

By eventually conjugating C_0 and A_F by the same element of G_F we may assume that $A_F = A_{0,F}$. Let $f_0, f_t, f_F, f_{t,F}$ be the concave functions associated respectively to C_0, tC_0, C_F and tC_F ; we have:

$$\begin{aligned} d_E(C_0, tC_0) &= 2 \sum_{\alpha \in \Phi^+} |f_t(\alpha) - f_0(\alpha)|; \\ d_F(C_F, tC_F) &= \sum_{\alpha \in \Phi^+} |f_{t,F}(\alpha) - f_F(\alpha)|. \end{aligned}$$

On the other hand, since $t \in T_F$, for every α , $f_t(\alpha) - f_0(\alpha)$ is an integer, and we deduce from this that $f_{t,F}(\alpha) - f_F(\alpha) = f_t(\alpha) - f_0(\alpha)$. The result follows immediately. \square

Lemma 7.3. *There exists an integer N_0 such that for every $g \in G_F$, we have $d_E(C, gC) \geq 2d_F(C_F, gC_F) - N_0$.*

Let g be an element of G_F , let \mathcal{A} be an apartment of \mathcal{B}_F containing both $R(C_F)$ and $R(gC_F)$, let T be the corresponding maximal F -split torus of G_F and let $N_G(T)$ be the normalizer of T in G ; we have $gC_F = nC_F$ for some element n of $N_G(T)_F$, hence g is of the form nh , with $h \in K_{C_F, F}$.

Let x be a special vertex of C_F ; we can write $n = tn_0$, with $t \in T$ and $n_0 \in K_{x,F}$. Set $C' = gC$, $C'' = n_0hC$ and $C_F'' = n_0C_F$; C_F'' also admits x as a vertex. Since n_0h always belongs to the open compact subgroup $K_{x,F}$ of G_F , the union of the n_0hC (resp. of the n_0C_F) is bounded, which implies that there exists an integer N'' (resp. N_F'') such that we always have $d_E(C, C'') \leq N''$ (resp.

$d_F(C_F, C_F'') \leq N_F''$). Moreover, according to lemma 7.2, setting $C_0'' = n_0 h C_0$ and $C_0' = t C_0'' = g C_0$, we have $d_E(C_0'', C_0') = 2d_F(C_F'', C_F')$. On the other hand, since $C'' = n_0 h C$ and $C_0'' = n_0 h C_0$, we have $d_E(C'', C_0'') = d_E(C, C_0)$; similarly, $d_E(C', C_0') = d_E(C, C_0)$. We finally obtain:

$$\begin{aligned} d_E(C, C') &\geq d_E(C', C'') - d_E(C'', C) \\ &\geq d_E(C_0', C_0'') - d_E(C', C_0') - d_E(C'', C_0'') - d_E(C'', C) \\ &\geq 2d_F(C_F', C_F'') - 2d_E(C, C_0) - N'' \\ &\geq 2d_F(C_F, C_F') - 2d_F(C_F, C_F'') - 2d_E(C, C_0) - N'' \\ &\geq 2d_F(C_F, C_F') - 2N_F'' - 2d_E(C, C_0) - N''. \end{aligned}$$

We thus can set $N_0 = 2N_F'' + 2d_E(C, C_0) + N''$; the lemma is now proved. \square

We can now prove the proposition. We can write:

$$\sum_{C \in O} |f(C)| = \frac{1}{[K_{C,F} : K_{C_F,F} \cap K_{C,F}]} \sum_{g \in G_F/K_{C,F}} |f(gC)|.$$

It is easy to check by induction that the number of chambers C_F'' of X_F whose retraction on $A_{0,F}$ relatively to C_F is some given chamber $C_{1,F}$ is $q^{d_F(C_F, C_{1,F})}$. By lemma 7.3, we obtain, W' being the affine Weyl group of G relative to T_0 :

$$\begin{aligned} \sum_{g \in G_F/K_{C,F}} |f(gC)| &\leq \sum_{g \in G_F/K_{C_F,F}} \frac{1}{q^{2d_F(C_F, gC_F) - N_0}} \\ &= \sum_{w \in W'} \frac{q^{d_F(C_F, wC_F)}}{q^{2d_F(C_F, wC_F) - N_0}} \\ &= \sum_{w \in W'} \frac{1}{q^{l(w) - N_0}}. \end{aligned}$$

By [14, section 3], the above sum converges for every $q > 1$. The result follows immediately. \square

7.2. The case A_d , d even. Now we prove theorem 1.1 when Φ is of type A_d , with $d = 2n$ being even, and q is large enough. First we have:

Proposition 7.4. *Assume Φ is of type A_d , d even. Then the Prasad character χ of F is trivial.*

Let ρ be the half-sum of the elements of Φ^+ . Write $\rho = \sum_{i=1}^d \lambda_i \alpha_i$, the α_i being the elements of Δ ; by [4, plate I, (VII)], we have:

$$\rho = \sum_{i=1}^d \frac{i(d+1-i)}{2} \alpha_i.$$

Hence λ_i is an integer for every i , and the proposition follows immediately from [10, lemma 3.1]. \square

Now define the set Ch_c of chambers of X_E as in corollary 5.3; by proposition 5.2, Ch_c is G_F -stable and G_F acts transitively on it. Set:

$$\lambda : f \in \mathcal{H}(X_E)^\infty \longmapsto \sum_{C \in Ch_c} f(C).$$

The linear form λ is well-defined by proposition 7.1, and obviously G_F -invariant. We want to prove that it is not identically zero on $\mathcal{H}(X_E)^\infty$.

By a slight abuse of notation, for every $C, C' \in Ch_c$, we write $d_F(C, C')$ for the combinatorial distance between the chambers of X_F whose geometric realizations contain respectively $R(C)$ and $R(C')$.

Let C be any element of Ch_E , and let I be the Iwahori subgroup of G_E fixing C . A well-known result about the Steinberg representation (see [19] for example) says that there exists a unique (up to a multiplicative constant) I -invariant element in the space of St_E , hence also in $\mathcal{H}(X_E)^\infty$. More precisely, set:

$$\phi_C : C' \in Ch_E \longmapsto (-q)^{-d_E(C, C')}.$$

It is easy to check that ϕ_C is I -invariant and satisfies the harmonicity condition. Hence every I -invariant element of $\mathcal{H}(X_E)^\infty$ is proportional to ϕ_C ; ϕ_C is called the (normalized) *Iwahori-spherical vector* of $\mathcal{H}(X_E)^\infty$ attached to C . Of course ϕ_C depends on C .

Now we prove the following proposition, from which theorem 1.1 follows immediately when G is of type A_{2n} :

Proposition 7.5. *Let C_0 be any element of Ch_c . Then ϕ_{C_0} is a test vector for λ . More precisely, we have $\lambda(\phi_{C_0}) = 1$.*

Let $C_{0,F}$ be the chamber of X_F whose geometric realization contains $R(C_0)$, let $C'_{0,F}$ be any chamber of X_F adjacent to $C_{0,F}$ and let C'_0 be the unique element of Ch_c whose geometric realization is contained in $R(C'_{0,F})$.

Let \mathcal{A} be an apartment of \mathcal{B}_F containing both $R(C_{0,F})$ and $R(C'_{0,F})$. Then \mathcal{A} also contains both $R(C)$ and $R(C')$, hence also every minimal gallery between them.

First we prove the following lemmas:

Lemma 7.6. *The combinatorial distance between C_0 and C'_0 is 3.*

Let A_E be the apartment of X_E whose geometric realization is \mathcal{A} and let C be a chamber of A_E adjacent to C_0 ; since by definition of Ch_c none of the walls of $R(C_0)$ is contained in a codimension 1 facet of \mathcal{B}_F , $R(C)$ is also contained in $R(C_{0,F})$, and since C'_0 is not an element of Ch_c , at least one of its walls has its geometric realization contained in $R(D)$, where D is a wall of $C_{0,F}$. On the other hand, since G_F is of type A_{2n} , the group of isomorphisms of \mathcal{B}_F which stabilize $C_{0,F}$ is of order $2n+1$ by [4, plate I], hence acts transitively on the set of its walls; we can then assume without loss of generality that D_F is the wall between $C_{0,F}$ and $C'_{0,F}$. Let C'' be the chamber of A_E which is separated from C by some wall

whose geometric realization is contained in $R(D_F)$; by symmetry, C' is adjacent to C'_0 . Hence (C_0, C, C', C'_0) is a gallery of length 3 between C_0 and C'_0 . On the other hand, every gallery $(C_0 = C, C_1, \dots, C_s = C')$ between C and C' contained in A must contain two chambers C_{i+1} and C_i separated by the hyperplane of A_E whose geometric realization contains $R(D_F)$, and the geometric realization of their common wall is then contained in \mathcal{B}_F ; since C_0 and C'_0 are both elements of Ch_c , they are both distinct from both C_i and C_{i+1} , and the length of any gallery between them is then at least 3. The result follows. \square

We deduce immediately from the lemma the following corollary:

Corollary 7.7. *Let \mathcal{H} be the hyperplane of \mathcal{A} containing $R(D_F)$. For every $C \in Ch_c$ whose geometric realization is contained in \mathcal{A} , $d_E(C'_0, C) - d_E(C_0, C)$ is contained in $\{-3, -1, 1, 3\}$, and is positive (resp. negative) if $R(C)$ is contained in the same half-apartment with respect to \mathcal{H} as $R(C_0)$ (resp. $R(C'_0)$).*

Now we examine more closely the structure of the subcomplex Ch_\emptyset .

Lemma 7.8. *There are exactly two chambers of A_E adjacent to C_0 and such that the geometric realization of one of their walls is contained in \mathcal{H} .*

Let H_E be the hyperplane of A_E whose geometric realization is \mathcal{H} . We already know that there exists at least one chamber satisfying these conditions, namely the chamber C of the gallery of length 3 between C_0 and C'_0 defined during the proof of lemma 7.6. Since every such chamber contains a wall of C_0 , its intersection with C_0 contains a facet D of H_E of codimension at most 2, and in fact of codimension exactly 2 since by hypothesis H_E does not contain any wall of C_0 . Since exactly two walls of C_0 contain D , there are also two chambers of A_E adjacent to C_0 and containing D .

Let C' be the unique chamber distinct from C satisfying these conditions; we now only have to prove that one of the walls of C' is contained in H_E . Let K_D be the connected fixator of D , and let \mathbb{G}_D be the quotient of K_D by its pro-unipotent radical; \mathbb{G}_D is then the group of k_E -points of a reductive group defined over k_E whose root system is of rank 2 and contained in a system of type A_{2n} , hence of type either A_1^2 or A_2 , and the combinatorial distance between two chambers containing D is equal to the combinatorial distance between the corresponding chambers in the spherical building of \mathbb{G}_D . If \mathbb{G}_D is of type A_1^2 , the combinatorial distance between C_0 and C'_0 can be at most 2, which contradicts lemma 7.6; hence \mathbb{G}_D must be of type A_2 . Since the order of its Weyl group is then 6, K_D contains exactly 6 Iwahori subgroups of G_E containing the maximal compact subgroup $K_{T,E}$ of T_E , where T is the maximal torus of G associated to A_E , or equivalently, D is contained in exactly 6 chambers of A_E . Out of these six chambers, exactly four admit as a wall some facet of maximal dimension of any given hyperplane of A_E containing D ; this is in particular true for H_E . On the other hand, C_0 is one of these six chambers, and by symmetry C'_0 must be another one. Since none of

these two admit any facet of maximal dimension of H_E as a wall, then C' must admit one and the lemma is proved. \square

Lemma 7.9. *Let C be a chamber of A_E adjacent to C_0 . There are exactly two walls of C whose geometric realizations are contained in walls of $R(C_F)$.*

Let C , C' , D and H_E be defined as in the previous lemma. Since C and C' are both adjacent to C_0 and all three of them belong to A_E , C and C' cannot be adjacent to each other, hence their intersection is D , which proves that the walls of C and C' contained in H_E are distinct. Hence by the previous lemma, the total number of walls of chambers of A_E adjacent to C_0 whose geometric realizations are contained in the walls of $R(C_F)$ is $2(2n+1)$. On the other hand, as we have already seen, the group of automorphisms of A_E stabilizing C_0 acts transitively on the set of its walls, hence also on the set of chambers of X_E adjacent to C_0 ; since its action obviously preserves the number of walls of C whose geometric realization is contained in walls of $R(C_F)$, that number must be two. \square

Let I_0 be the Iwahori subgroup of G_E fixing C_0 ; we have the following lemma:

Lemma 7.10. *The number of elements of Ch_c which are conjugated to C by some element of I_0 is $q^{d_F(C_0, C)}$.*

By [5, lemma 4.2] and an obvious induction, it is enough to prove that two elements of Ch_c are conjugated by an element of I_0 if and only if they are conjugated by an element of $I_{0,F} = I_0 \cap G_F$. Let C'' be an element of Ch_c conjugated to C by some element of I_0 , and let C_F (resp. C_F'') be the chamber of X_F whose geometric realization contains $R(C)$ (resp. $R(C'')$). There exists then an element of $I_{0,F} = I_0 \cap G_F$ sending C_F to C_F'' , and by unicity of the central chamber in the geometric realization of C_F (resp. C_F''), that element must send C on C'' . The other implication being obvious, the lemma is proved. \square

Now we prove proposition 7.5. Let C_0 be the only element of Ch_c whose geometric realization is contained in $R(C_{0,F})$, let C be any element of Ch_c , set $d = d_E(C_0, C)$, and let C_1 be a chamber of $A_{0,E}$ adjacent to C and such that H_E contains a wall D_1 of C_1 . First we assume that C satisfies the following property:

(P1): There exists a minimal gallery of the form $(C_0, C_1, \dots, C_\delta = C)$,

and that C_0 and C are in the same half-space of $A_{0,E}$ with respect to H_E . Let C'_1 be the other chamber of \mathcal{B}_E admitting D_1 as a wall. Then $(C'_0, C'_1, C_1, \dots, C_\delta)$ is a minimal gallery of length $\delta+1$, from which we deduce by symmetry that if C' is the image of C by the orthogonal reflection with respect to H_E , $d(C_0, C') = \delta+1$. Hence we have $\phi_{C_0}(C) = (-q)^{-\delta}$ and $\phi_{C_0}(C') = (-q)^{-\delta-1}$.

On the other hand, by the same reasoning, if we set $\delta' = d_F(C_0, C')$, we have $d_F(C_0, C') = \delta' + 1$. From lemma 7.10, we deduce that the sum of the $f(C'')$, when C'' runs through the set of conjugates of C (resp. C') by elements of $I_{0,F}$ is $\frac{q^{d_F(C_0, C)}}{(-q)^\delta}$ (resp. $\frac{q^{d_F(C_0, C')}}{(-q)^{\delta+1}} = \frac{q^{d_F(C_0, C)+1}}{(-q)^{\delta+1}}$). Since these two values are opposite to each other, their sum is zero. Since this is true for every C satisfying **(P1)** and on the same side of H_E as C_0 , we obtain the following lemma:

Lemma 7.11. *The sum of the $\phi_{C_0}(C)$, when C runs through the set of all conjugates by elements of $I_{0,F}$ of all elements of Ch_c satisfying **(P1)** and on the same side of H_E as C_0 and of their images by the reflection with respect to H_E , is zero.*

From now on, we denote by Ch_{c,C_1} the set of such C .

Now let C_1'' be the other chamber adjacent to C and such that H contains a wall D_1'' of C_1'' ; we have:

Lemma 7.12. *Let C be any element of Ch_c contained in $A_{0,E}$. The following conditions are equivalent:*

- C is either a chamber satisfying **(P1)** and on the same side of H_E as C_0 or the image by the reflection with respect to H_E of such a chamber;
- there exist minimal galleries between C_0 and C containing C_1 but none containing C_1'' .

Let D_0 (resp. D'_0) be the wall separating C_0 from C_1 (resp. C'_1), and let H_0 (resp. H'_0) be the hyperplane of A_E containing it. A chamber C of A_E satisfies the second condition if and only if it is separated from C_0 by H_0 but not by H'_0 . On the other hand, since H_0 , H'_0 and H are the only three hyperplanes of A_E containing $D_0 \cap D'_0$, H'_0 must be the image of H_0 by the orthogonal reflection with respect to H . Both conditions are then equivalent to: $R(C)$ is contained either in the connected component of $R(A_E) - (R(H_E) \cup R(H_0) \cup R(H'_0))$ containing C_1 or in its image by the orthogonal reflection with respect to $R(H_E)$. The lemma follows immediately. \square

On the other hand, since H_0 and H'_0 both contain walls of C_0 and are not perpendicular to each other, they correspond to consecutive roots in the extended Dynkin diagram of Φ . Since, Φ being of type A_{2n} , its extended Dynkin diagram is a cycle, we can label the hyperplanes $H_{0,1}, \dots, H_{0,2n+1}$ containing walls of C_0 in such a way that for every i , with $H'_{0,i}$ being defined relatively to $H_{0,i}$ the same way as H'_0 is defined relatively to H_0 , we have $H'_{0,i} = H_{0,i+1}$ (the indices being taken modulo $2n+1$). More precisely, for every i , let $C_{1,i}$ be the chamber of A_E separated from C_0 by $H_{0,i}$, let D_i be their common wall and let $D_{F,i}$ be the wall of C_F whose geometric realization contains D_i . Let $C'_{1,i}$ be the unique chamber of A_E neighboring C_0 , containing a wall whose geometric realization is contained in $D_{F,i}$ and distinct from $C_{1,i}$; such a chamber exists and is unique by lemma 7.8. Let $H'_{0,i}$ be the hyperplane of A_E separating C_0 from $C'_{1,i}$, we then have $H'_{0,i} = H_{0,i+1}$.

Let also A_F be the apartment of X_F whose geometric realization is \mathcal{A} , and for every i , let H_i be the hyperplane of A_E whose geometric realization contains $R(D_{F,i})$, let $C'_{F,i}$ be the chamber of A_F separated from C_F by $D_{F,i}$ and let $C'_{0,i}$ be the unique element of Ch_c whose geometric realization is contained in $C'_{F,i}$.

Let now C be any element of Ch_c contained in A_E and different from C_0 . Let I_C be the subset of the elements $i \in \mathbb{Z}/(2n+1)\mathbb{Z}$ such that C is separated from C_0 by $H_{0,i}$; since $C \neq C_0$, I_C is nonempty, and since the closure of $C \cup C_0$ must

contain at least one wall of C_0 , I_C is not the whole set $\mathbb{Z}/(2n+1)\mathbb{Z}$ either. Hence the set I'_C of elements i of $\mathbb{Z}/(2n+1)\mathbb{Z}$ such that $i \in I_C$ and $i+1 \notin I_C$ is nonempty.

For every i , set $Ch_{c,i} = Ch_{c,C_{1,i}}$, and for every $I' \subset \mathbb{Z}/(2n+1)\mathbb{Z}$, set $Ch_{c,I'} = \bigcap_{i \in I'} Ch_{c,i}$; for every I' and every $C \in Ch_c$ contained in A_E , we have $C \in Ch_{c,I'}$ if and only if $I' \subset I'_C$, and we thus obtain:

$$\sum_{C \in Ch_c} \phi_{C_0}(C) = \phi_{C_0}(C_0) + \sum_{I' \subset \mathbb{Z}/(2n+1)\mathbb{Z}, I' \neq \emptyset} (-1)^{\#(I')+1} \sum_{C \in Ch_{c,I'}} \phi_{C_0}(C).$$

Since $\phi_{C_0}(C_0) = 1$, to prove proposition 7.5, it is now enough to prove the following result:

Proposition 7.13. *For every nonempty subset I' of $\{1, \dots, 2n+1\}$, we have $\sum_{C \in Ch_{c,I'}} \phi_{C_0}(C) = 0$.*

We already know by lemma 7.11 that the assertion of the proposition holds when I' is a singleton; we now have to prove it in the other cases.

First we remark that since for every C and for every $i \in I'_C$, i belongs to I_C but $i+1$ does not, a necessary condition for $Ch_{c,I'}$ to be nonempty is that I' does not contain two consecutive elements of $\mathbb{Z}/(2n+1)\mathbb{Z}$. In the sequel, we assume that I' satisfies that condition.

For every $i \in \mathbb{Z}/(2n+1)\mathbb{Z}$, let $|i|$ be the distance between i and 0 in the cyclic group: for example, $|1|$ is 1, and $|2n|$ is also 1. We have:

Lemma 7.14. *Let $i, j \in I'$ be such that $|i - j| \geq 3$. Then all three of $H_{0,i}$, $H'_{0,i}$, H_i are orthogonal to all three of $H_{0,j}$, $H'_{0,j}$, H_j .*

Let $\varepsilon_1, \dots, \varepsilon_d$ be elements of $X^*(T) \otimes \mathbb{Q}$ defined as in [4, plate I]. Assume the ε_i are numbered in such a way that for every i , $H_{0,i}$ corresponds to the roots $\pm(\varepsilon_i - \varepsilon_{i+1})$. Then $H'_{0,i}$ (resp. H_i) corresponds to the roots $\pm(\varepsilon_{i+1} - \varepsilon_{i+2})$ (resp. $\pm(\varepsilon_i - \varepsilon_{i+2})$). The lemma follows immediately. \square

This lemma proves that the union of the elements of the intersection $Ch_{c,\{i,j\}} = Ch_{c,i} \cap Ch_{c,j}$ whose geometric realization is contained in \mathcal{A} is symmetrical with respect to H_i (or H_j , for that matter); we deduce from this, using the same reasoning as for $Ch_{c,i}$ in lemma 7.11, that $\sum_{C \in Ch_{c,\{i,j\}}} f(C) = 0$. More generally, we divide I' into segments the following way: $I' = I'_1 \cup \dots \cup I'_r$, where every I'_k is of the form $\{i, i+2, \dots, i+2(l_k-1)\}$, l_k being the length of the segment, and if $i \in I'_k$ and $j \in I'_l$ with $k \neq l$, then $|i - j| \geq 3$; such a partition of I' into segments exists since I' cannot contain two consecutive elements of $\mathbb{Z}/(2n+1)\mathbb{Z}$, and is obviously unique up to permutation of the segments. We then prove in a similar manner as for $I' = \{i, j\}$ that we have $\sum_{C \in Ch_{c,I'}} f(C) = 0$ as soon as one of the I'_k is a singleton.

Consider now the case where I' is a single segment of length $l > 1$, say for example $I' = \{1, 3, \dots, 2l-1\}$. Then if C is an element of $Ch_{c,I'}$ contained in \mathcal{A} ,

the concave function f_C associated to C (normalized by taking C_0 as the standard Iwahori) must satisfy the following conditions:

- for every $i \in \{0, \dots, l-1\}$, $f_C(\varepsilon_{1+2i} - \varepsilon_{2+2i}) \geq \frac{1}{2}$;
- for every $i \in \{0, \dots, l-1\}$, $f_C(\varepsilon_{2+2i} - \varepsilon_{3+2i}) \leq 0$.

Since $f_C(\alpha) + f_C(-\alpha) = 1$ for every $\alpha \in \Phi$, we obtain:

- for every $i \in \{0, \dots, l-1\}$, $f_C(\varepsilon_{2+2i} - \varepsilon_{1+2i}) \leq 0$;
- for every $i \in \{0, \dots, l-1\}$, $f_C(\varepsilon_{3+2i} - \varepsilon_{2+2i}) \geq \frac{1}{2}$.

We can associate to C the $(l+1) \times l$ matrix $M = (m_{ij})$ defined the following way: for every $i \in \{0, \dots, l\}$ and every $j \in \{1, \dots, l\}$, $m_{ij} = 1$ (resp. $m_{ij} = 0$) if $f_C(\varepsilon_{2j} - \varepsilon_{1+2i}) \geq \frac{1}{2}$ (resp. ≤ 0). For every M , let $Ch_{c,I',M}$ be the set of $C' \in Ch_{c,I'}$ which are conjugated by an element of I_F to some chamber contained in A_E whose associated matrix is M ; we now prove that for every M , we have $\sum_{C \in Ch_{c,I',M}} f(C) = 0$.

We first investigate the conditions for $Ch_{c,I',M}$ to be nonempty. From the above conditions we see that we must have $m_{i-1,i} = m_{ii} = 0$ for every i . We now prove the following lemma:

Lemma 7.15. *Assume there exist i, i', j, j' such that $m_{ij} = m_{i'j'} = 1$ and $m_{ij'} = m_{i'j} = 0$. Then $Ch_{c,I',M}$ is empty.*

Let C be an element of $Ch_{c,I',M}$ contained in A_E . In terms of concave functions, the assertion of the lemma translates into: $f_C(\varepsilon_{2j} - \varepsilon_{1+2i}), f_C(\varepsilon_{2j'} - \varepsilon_{1+2i'}) \geq \frac{1}{2}$ and $f_C(\varepsilon_{2j} - \varepsilon_{1+2i'}), f_C(\varepsilon_{2j'} - \varepsilon_{1+2i}) \leq 0$. We deduce from this that we have $f_C(\varepsilon_{1+2i} - \varepsilon_{2j}) \leq 0$ and $f_C(\varepsilon_{1+2i'} - \varepsilon_{2j'}) \leq 0$, hence by concavity:

$$f_C(\varepsilon_{1+2i} - \varepsilon_{1+2i'}) \leq f_C(\varepsilon_{1+2i} - \varepsilon_{2j}) + f_C(\varepsilon_{2j} - \varepsilon_{1+2i'}) \leq 0,$$

$$f_C(\varepsilon_{1+2i'} - \varepsilon_{1+2i}) \leq f_C(\varepsilon_{1+2i'} - \varepsilon_{2j'}) + f_C(\varepsilon_{2j'} - \varepsilon_{1+2i}) \leq 0.$$

On the other hand, since C is a chamber, we must have $f_C(\varepsilon_{1+2i} - \varepsilon_{1+2i'}) + f_C(\varepsilon_{1+2i'} - \varepsilon_{1+2i}) = \frac{1}{2}$, which is impossible given the above inequalities. Hence $Ch_{c,I',M}$ must be empty and the lemma is proved. \square

From now on we assume that M is such that $Ch_{c,I',M}$ is nonempty.

Corollary 7.16. *For every i , let Z_i be the set of indices j such that $m_{ij} = 0$. Then for every i, i' , we have either $Z_i \subset Z_{i'}$ or $Z_{i'} \subset Z_i$.*

Assume there exist j, j' such that $j \in Z_{i'} - Z_i$ and $j' \in Z_i - Z_{i'}$. Then i, i', j, j' satisfy the conditions of the previous lemma, and M cannot then be nonempty. \square

Using this corollary, we define a total preorder on $\{0, \dots, l\}$ by $i \leq_M i'$ if and only if $Z_i \subset Z_{i'}$.

Lemma 7.17. *Let i be a maximal element for that preorder. Then Z_i is the full set $\{1, \dots, l\}$.*

As we have already seen, for every $j \in \{1, \dots, n\}$, $m_{jj} = 0$, hence $j \in Z_j \subset Z_i$.
 \square

Lemma 7.18. *There exists an $i \in \{0, \dots, l\}$ such that both i and $i - 1$ are maximal for the order \leq_M .*

Let i_0 be any maximal element of $\{0, \dots, l\}$ for \leq_M . If either $i_0 - 1$ or $i_0 + 1$ is maximal, there is nothing to prove; assume that none of them is maximal. Let j be an element of $Z_{i_0} - Z_{i_0+1}$; since $m_{j-1,j} = m_{jj} = 0$, j belongs to both Z_j and Z_{j-1} , and we then have $i_0 + 1 <_M j$ and $i_0 + 1 <_M j - 1$. If both j and $j - 1$ are maximal, the lemma is proved, if either j or $j - 1$ is not maximal, assuming for example j is not, we now consider an index k not belonging to Z_j and we use the same reasoning as above to obtain that $j <_M k$ and $j <_M k - 1$; since our set of indices is finite, after a finite number of iterations we must reach an i such that both i and $i - 1$ are maximal, as desired. \square

Corollary 7.19. *Assume i is such that both i and $i - 1$ are maximal for $<_M$. Then the set of chambers in $Ch_{c,I',M}$ contained in A_E is symmetrical with respect to H_{2i-1} .*

It is easy to see that for every i , replacing a chamber C by its image by the symmetry with respect to H_{2i-1} is equivalent to switching the columns $i - 1$ and i in M . When $i - 1$ and i are both maximal for \leq_M , these columns are identical, hence M is preserved. \square

We can now prove that $\sum_{C \in Ch_{c,I',M}} f(C) = 0$ the same way as when I' is a singleton: let i be an integer associated to M by lemma 7.18, and let C, C' be the two chambers adjacent to C_0 and such that the geometric realization of one of their walls is contained in the geometric realization of H_{2i-1} (these chambers exist by lemma 7.8). With the help of corollary 7.19, we can now, by the same reasoning as in lemma 7.11, obtain the desired result. Since this is true for every M , we obtain that $\sum_{C \in Ch_{c,I'}} f(C) = 0$ when I' is a single segment.

We finally use, with the help of lemma 7.14, the same reasoning applied to any one of the segments of I' to prove that $\sum_{C \in Ch_{c,I'}} f(C) = 0$ in the general case.
 \square

Since by that proposition, $\sum_{C \in Ch_c} \phi_{C_0}(C) = \phi_{C_0}(C_0) \neq 0$, ϕ_{C_0} is a test vector for λ , and theorem 1.1 is now proved when G is of type A_{2n} and q is large enough.
 \square

Remark: in [5], where λ is defined in a similar way as in this subsection, since E/F is unramified, the sum defining λ converges because at every step, there are $q_E = q^2$ times more chambers on the building itself, which implies that for every $f \in \mathcal{H}(X_E)^\infty$, for chambers C' located far away enough from the origin, at every step, $f(C')$ is divided by q^2 and we only have q times more chambers to consider (see [5, lemmas 4.3 and 4.4]). In the tamely ramified case, for the groups of type A_{2n} we are considering here, there are only q times more chambers on the

building itself when the distance increases by 1, but at every step, the distance increases by 2 on average (lemma 7.2), and the sum converges for that reason. We will see in the sequel that a similar argument applies to other types of groups as well.

7.3. The other cases. In this subsection, we assume that Φ is not of type A_{2n} for any n . Let Σ_a be a subset of Φ satisfying the conditions of proposition 5.11; we will prove that there exists a linear form λ on $\mathcal{H}(X_E)^\infty$ with support in the F -anisotropy class Ch_a of Ch_E corresponding to Σ_a and a test vector $f \in \mathcal{H}(X_E)^\infty$ such that $\lambda(f) = 0$. Note that this time, our test vector will not be Iwahori-spherical.

Let T be a E -split maximal F -torus of G of F -anisotropy class Σ_a , let A be the Γ -stable apartment of X_E associated to T and let D be a facet of A^Γ of maximal dimension. We assume that D and Σ_a have also been chosen in such a way that either proposition 5.13 or (in cases A_d , d odd, D_d , d odd and E_6) proposition 5.14 is satisfied.

As in the previous section, we denote by Φ_D the smallest Levi subsystem of Φ containing Σ_a ; Φ_D is also the root system of $K_{D,E}/K_{D,E}^0$, where $K_{D,E}^0$ is the pro-unipotent radical of $K_{D,E}$.

Let $\mathcal{H}(Ch_D)$ be the space of harmonic cochains on Ch_D . First we prove that there actually exists an element of $\mathcal{H}(Ch_D)$ with support in $Ch_{D,a}$ which is stable by $K_A \cap G_{F,der}$ and not identically zero on Ch_D . Let ϕ_D be the function on Ch_D defined the following way:

- the support of ϕ_D is $Ch_{D,a}$;
- $\phi_D(C(1, \dots, 1)) = 1$, and for every $\lambda_1, \dots, \lambda_r \in k_F^*$, $\phi_D(C(\lambda_1, \dots, \lambda_r))$ is either 1 or -1 , its values being chosen in such a way that, $f = \phi_D$ being viewed as a function on $H^1(\Gamma, K_{T \cap L_{E,der}})$, the relations of proposition 6.22 are all satisfied;
- ϕ_D is $K_D \cap G_{F,der}$ -stable.

First we check that the definition is consistent. The map $(\lambda_1, \dots, \lambda_r) \mapsto \phi_D(C(\lambda_1, \dots, \lambda_r))$ being a group morphism from $(k_F^*)^r$ to $\{\pm 1\}$, it is enough to prove the following lemma:

Lemma 7.20. *For every $g \in K_D \cap G_{F,der}$ such that $C' = gC(1, \dots, 1)$ is of the form $C(\lambda_1, \dots, \lambda_r)$, we have $\phi_D(C') = 1$.*

First we prove that we can assume g is an element of $T_0 \cap G_{F,der}$. Let F' be the unique quadratic unramified extension of F ; we deduce from lemma 6.14 that there exists $t \in L_{F',der}$ such that $tC' = C$, and t obviously must belong to $K_{T_0, F'}$. Set $g' = gt$; g' is then an element of $K_{D, F'} \subset G_{F'}$ such that $g'C = C$. On the other hand, such an element must satisfy $g'\gamma(C) = \gamma(C)$ as well, hence is contained in $K_{C \cap \gamma(C), F'} = K_{T, F'} K_{D, F'}^0 \subset L_{F'} \cap K_{D, F'}$, and since $f_D(\beta) \in \frac{1}{2} + \mathbb{Z}$ for every $\beta \in \Sigma_a$, we have $U_\beta \cap K_{D, F'} \subset K_{D, F'}^0$ for every β , from which we deduce

that $L_{F'} \cap K_{D,F'} \subset T_{0,F'} K_{D,F'}^0$. Hence we can assume $g' \in T_0$, which implies that $g \in T_0$ as well.

We now assume g is an element of $T_0 \cap G_{F,der}$, and even that g is of the form $\alpha^\vee(c)$, with α being a simple root in Φ^+ and c being an element of \mathcal{O}_F^* which is not a square.

First we remark that when $\alpha \in -\Sigma_a$, say $\alpha = \beta_1$ for example, we have:

$$\alpha^\vee(c)C(1, \dots, 1) = C(c^2, 1, \dots, 1)$$

and since we obviously have $\phi_D(C(c^2, 1, \dots, 1)) = 1$, the result follows.

Now we deal with the other simple roots with the help of a case-by-case analysis. Notations are the same as in proposition 6.22.

- Assume Φ is of type A_d , with $d = 2n - 1$ being odd. Then the simple roots α_{2i-1} , $i = 1, \dots, n$, are all contained in $-\Sigma_a$, and when i is even, for every j , setting $\beta_j = \alpha_{2j-1}$, $\langle \beta_j, \alpha_i^\vee \rangle$ is -1 if j is either $\frac{i}{2}$ or $\frac{i}{2} + 1$, and 0 in the other cases; we then have:

$$\alpha_i^\vee(c)C(1, \dots, 1) = C(1, \dots, c^{-1}, c^{-1}, \dots, 1),$$

the c^{-1} being in j -th and $j + 1$ -th position; hence in $H^1(\Gamma, K_{T \cap L_{E,der}})$, we obtain $\alpha_i^\vee(c) = e_j e_{j+1}$. By proposition 6.22, for every $\lambda_1, \dots, \lambda_n$, we have:

$$\phi_D(C(\lambda_1, \dots, \lambda_n)) = (-1)^s \phi_D(C(1, \dots, 1)) = (-1)^s,$$

where s is the number of λ_i which are not squares; the result follows immediately.

- Assume Φ is of type B_d . Then the simple roots α_i , with i odd, are all contained in $-\Sigma_a$. On the other hand, when i is even and strictly smaller than d , α_i has already been dealt with in proposition 6.21. I will explicit what it means in this case, the other cases being treated similarly. By the relations we have found in proposition 6.21, for every such i , we have, in $H^1(\Gamma, K_{T \cap L_{E,der}})$, $\alpha_i^\vee(c)1 = e_{i-1}e_i e_{i+1}e_{i+2}$, and we deduce immediately from proposition 6.22 that $\phi_D(\alpha_i^\vee(c)C(1, \dots, 1)) = 1$; which is the expected result. When d is odd, the result is now proved, and when d is even, it only remains to consider α_d^\vee . We have $\langle \beta_i, \alpha_d^\vee \rangle = -2$ if i is either $d - 1$ or d and 0 in the other cases, hence:

$$\alpha_d^\vee(c)C(1, \dots, 1) = C(1, \dots, 1, c^{-2}, c^{-2}).$$

The result follows immediately.

- Assume Φ is of type C_d . The only simple root contained in $-\Sigma_a$ is then α_d , and for every $i < d$, $\langle \alpha_i, \beta_j \rangle$ is -2 if j is either i or $i + 1$ and 0 else, hence we have:

$$\alpha_i^\vee(c)C(1, \dots, 1) = C(1, \dots, c^{-2}, c^{-2}, \dots, 1).$$

The result follows.

- Assume Φ is of type D_d . The simple roots contained in $-\Sigma_a$ are the α_i , with $d - i$ odd, and α_d . The α_i , with $d - i$ even and $1 < i < d$, have already been dealt with in proposition 6.21, and when d is odd, we have in $H^1(\Gamma, K_{T \cap L_{E,der}})$, by proposition 6.21, $\alpha_1^\vee(c)1 = e_1 e_2$. On the other hand, by proposition 6.22, we have:

$$\phi_D(e_1 e_2) = \phi_D(1).$$

The result follows.

- Assume Φ is of type E_6 . The simple roots contained in $-\Sigma_a$ are α_2 , α_3 and α_5 , and α_4 has already been dealt with in proposition 6.21. Now consider α_1 ; we have:

$$\alpha_1^\vee(c)C(1, 1, 1, 1) = C(c^{-1}, c^{-1}, 1, 1).$$

hence in $H^1(\Gamma, K_{T \cap L_{E,der}})$, we have $\alpha_1^\vee(c)1 = e_1 e_2$. On the other hand, by proposition 6.22, we have $\phi_D(e_1 e_2) = \phi_D(1)$. The case of α_6 being symmetrical, the result follows.

- Assume Φ is of type F_4 . The only simple root contained in $-\Sigma_a$ is α_2 , and α_1 has already been dealt with in proposition 6.21. On the other hand, we have:

$$\begin{aligned} \alpha_3^\vee(c)C(1, 1, 1, 1) &= C(1, 1, c^2, c^{-2}); \\ \alpha_4^\vee(c)C(1, 1, 1, 1) &= C(1, c^2, c^{-2}, 1). \end{aligned}$$

The result follows immediately.

- In the three remaining cases (E_7 , E_8 and G_2), every simple root either belongs to $-\Sigma_a$ or has been dealt with in proposition 6.21; these cases then follow immediately from that proposition.

The lemma is now proved. \square

Now we check that ϕ_D satisfies the harmonicity condition.

Proposition 7.21. *Let D_1 be any codimension 1 facet of X_E containing D ; the sum of the values of ϕ_D on the chambers containing D_1 is zero.*

If D_1 is not contained in any element of $Ch_{D,a}$, the harmonicity condition is trivially satisfied; we can thus assume that D_1 is contained in some $C \in Ch_{D,a}$, and even, by eventually conjugating it, in some $C \in Ch_{D,a,L,C_0}$. Let D' be the unique codimension 1 facet of C_0 of the same type as D_1 , or in other words the only one which is $G_{E,der}$ -conjugated to D_1 . Let α be the corresponding simple root in Φ_D^+ ; assume first there exists a conjugate Σ' of Σ_a in Φ_D containing α . Since α is a simple root, by definition of f_{C_0} , we have $f_{C_0}(-\alpha) = 1 \in \mathbb{Z}$.

Let $\Phi_D'^+$ be any set of positive roots of Φ_D such that α is a simple root in $\Phi_D'^+$, and let C'_0 be the unique chamber of $A_{0,E}$ containing D such that $-\Phi_D'^+$ is the set of roots of the Borel subgroup of $K_{D,E}/K_{D,E}^0$ corresponding to it. For every $\lambda_1, \dots, \lambda_r \in \mathcal{O}_E^*$, we define the chamber $C''(\lambda_1, \dots, \lambda_r) \in Ch_{D,a,L,C'_0}$ in a similar way as $C(\lambda_1, \dots, \lambda_r)$. Since $f_{C_0}(-\alpha)$ is an integer, by proposition 6.11, there exist $\lambda_1, \dots, \lambda_r$ such that $C''(\lambda_1, \dots, \lambda_r)$ is $K_D \cap G_{F,der}$ -conjugated to C .

Let D'_1 be the codimension 1 facet of $C'(\lambda_1, \dots, \lambda_n)$ of the same type as D' ; D'_1 and D' are then $G_{F,der}$ -conjugates, which implies that every chamber of X_E containing D'_1 is then $G_{F,der}$ -conjugated to some chamber of X_E containing D' ; and that these conjugations induce a bijection between these two set of chambers; the harmonicity condition for the chambers containing D'_1 , which follows from lemma 6.17, then implies the harmonicity condition for those containing D_1 .

On the other hand, two roots of the same length are always conjugates, hence the condition on α holds as soon as Σ_a contains roots of every length. This is trivially true when Φ is simply-laced, and we see from proposition 5.13 that it is also true for types B_d , d odd, and G_2 .

Assume now we are in one of the remaining cases (B_d with d even, C_d for any d and F_4); Σ_a then contains only long roots, and the above proof still works when α is long. Assume now α is short, and let β be a long root belonging to Φ^+ and not orthogonal to α ; α and β then generate a subsystem of Φ of type B_2 , hence either $\beta + 2\alpha$ or $\beta - 2\alpha$ is also a long root, and that root must also belong to Φ^+ (it is obvious for $\beta + 2\alpha$; for $\beta - 2\alpha$, as in lemma 6.16, it comes from the fact that β contains at least one simple root different from α in its decomposition, hence $\beta - 2\alpha$ cannot be negative). In both cases, α is the half-difference of two long roots belonging to Φ^+ , and we are then in the situation of lemma 6.18; the harmonicity condition for D_1 then follows immediately from the expression (2) in the proof of that lemma. \square

Now we check that ϕ_D is compatible with the Prasad character χ , or in other words that $\phi_D(gC) = \chi(g)\phi_D(C)$ for every $g \in G_F$ and every $C \in Ch_D$. Let $K_{T_0,F}$ be the maximal compact subgroup of $(T_0)_F$ and let $X_{T_0,F}$ be the subgroup of $(T_0)_F$ generated by the $\xi(\varpi_F)$, where ξ runs over the one-parameter subgroups of T_0 .

Remember that we have a decomposition $G_F = G_{F,der}K_{T_0,F}X_{T_0,F}$, and also that the ϖ_F we have chosen is the norm of some element of E . The character χ is trivial on $G_{F,der}$ and on $X_{T_0,F}$; the compatibility of ϕ_D with χ is then an immediate consequence of the following proposition:

Proposition 7.22. *Let t be any element of the maximal compact subgroup K_{T_0} of $T_{0,F}$. Then for every $C \in Ch_D$ and every $f \in \mathcal{H}(X_E)^{G_{F,der}}$, we have $f(tC) = \chi(t)f(C)$.*

Let C be any element of Ch_D . If C does not belong to $Ch_{D,a}$, then neither does tC and we then have $f(tC) = \chi(t)f(C) = 0$. We thus may assume that $f \in Ch_{D,a}$, and by eventually conjugating it, we can even assume that C belongs to Ch_{D,a,L,C_0} .

We already know from lemma 7.20 that if $t \in G_{F,der}$, $f(tC) = f(C) = \chi(t)f(C)$ since $\chi(t) = 1$. Moreover, if t is a square, then its image in $H^1(\Gamma, K_{T \cap L_{E,der},F})$ is a square too, hence trivial by proposition 6.13, and since χ is quadratic, $\chi(t)$ is trivial too. Hence we only have to prove the result when t belongs to some set

of representatives in $T_0 \cap G_F$ of some set of generators of the finite abelian group Y/Y^2 , where $Y = (T_0 \cap G_F)/(T_0 \cap G_{F,der})$.

Let ρ be the half-sum of the elements of Φ^+ ; by [4, §1, proposition 29] and [10, lemma 3.1], for every $t \in T_0 \cap G_F$, $\chi(t) = 1$ if and only if $2\rho(t)$ is the norm of some element of E^* . We refer to [4, plates I to IX] for the expressions of Y and ρ we use during the case-by-case analysis below. In the sequel, once again, c is an element of \mathcal{O}_F^* which is not a square.

Note first that the cases E_8 , F_4 and G_2 are trivial since we then have $G_F = G_{F,der}$. We examine the other cases.

- Assume Φ is of type A_{2n-1} . Then Y is cyclic of order $2n$, and with a slight abuse of notation, the element $t = \text{Diag}(c, 1, \dots, 1)$ of $GL_{2n}(F)$ is, for any choice of c , a representative of the unique nontrivial element of Y/Y^2 , hence can be used to compare two quadratic characters of Y . Since $\beta_1 = \alpha_1$, for every $h \in H^1(\Gamma, K_{T \cap L_{E,der}, F})$ and every $C \in Ch(h)$, by proposition 6.21, the chamber tC belongs to $Ch(e_1 h)$, and we deduce from proposition 6.22 that $\phi_D(tC) = -\phi_D(C)$. On the other hand, we have $2\rho = \sum_{i=1}^d i(d+1-i)\alpha_i$, hence $2\rho(t) = c^d$, hence $\chi(t) = (-1)^d$. We thus obtain $\phi_D(tC) = \chi(t)\phi_D(C)$, as desired.
- Assume now Φ is of type B_d ; Y is then of order 2 and its nontrivial element admits $t = \text{Diag}(c, 1, \dots, 1, c^{-1}) \in GSO'_{2d+1}(F)$, where GSO'_{2d+1} is the split form of GSO_{2d+1} , as a representative. We denote by n the largest integer such that $2n \leq d$.

By proposition 6.21, for every $h \in H^1(\Gamma, K_{T \cap L_{E,der}, F})$ and every $C \in Ch(h)$, tC belongs to $Ch(e_1 e_2 h)$, hence by proposition 6.22, $\phi_D(tC) = -\phi_D(C)$. On the other hand, we have $2\rho = \sum_{i=1}^d i(2d-i)\alpha_i$; we then obtain $2\rho(t) = c^{2d-1}$, hence $\chi(t) = -1$ and the result follows.

- Assume Φ is of type C_d ; Y is then of order 2, and with a slight abuse of notation, its nontrivial element admits $t = \text{Diag}(c, \dots, c, 1, \dots, 1) \in GSp_{2d}(F)$ as a representative. By proposition 6.21, for every element h of $H^1(\Gamma, K_{T \cap L_{E,der}, F})$ and every $C \in Ch(h)$, tC belongs to $Ch(e_1 \dots e_d h)$, hence by proposition 6.22, we have:

$$\begin{aligned} \phi_D(tC) &= \left(\prod_{i=1}^d (-1)^{d+1-i} \right) \phi_D(C) \\ &= \left(\prod_{i=1}^d (-1)^i \right) \phi_D(C) = (-1)^{\frac{d(d+1)}{2}} \phi_D(C). \end{aligned}$$

On the other hand, we have:

$$2\rho = \sum_{i=1}^{d-1} i(2d+1-i)\alpha_i + \frac{d(d+1)}{2}\alpha_d;$$

we then obtain $2\rho(t) = c^{\frac{d(d+1)}{2}}$, hence $\chi(t) = (-1)^{\frac{d(d+1)}{2}}$. We finally get $\phi_D(tC) = \chi(t)\phi_D(C)$ once again.

- Assume now Φ is of type D_d . When d is even, Y is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and is generated by the elements admitting respectively $t = \text{Diag}(c, 1, \dots, 1, c^{-1})$ and $t' = \text{Diag}(c, \dots, c, 1, \dots, 1)$, both belonging to $GSO'_{2d}(F)$, as representatives; when d is odd, Y is cyclic of order 4 and one of its generators admits t' as a representative. In both cases, we denote by n the largest integer such that $2n \leq d$.

When d is even, by proposition 6.21, for every $h \in H^1(\Gamma, K_{T \cap L_{E, \text{der}}, F})$ and every $C \in Ch(h)$, tC belongs to $Ch(e_1 e_2 h)$, hence by proposition 6.22, $\phi_D(tC) = \phi_D(C)$. On the other hand, we have:

$$2\rho = \sum_{i=1}^{d-2} i(2d-1-i)\alpha_i + \frac{d(d-1)}{2}(\alpha_{d-1} + \alpha_d);$$

hence $2\rho(t) = c^{2d-2}$, from which we obtain that $\chi(t) = 1$ and that $\phi_D(tC) = \chi(t)\phi_D(C)$, as desired.

Now we consider t' , d being either odd or even. By proposition 6.21, we have $t'Ch(h) = Ch(e_1 e_3 \dots e_{2n-1} h)$, hence by proposition 6.22, $\phi_D(t'C) = (-1)^n \phi_D(C)$; on the other hand, using the same expression as above for 2ρ , we obtain $\chi(t') = (-1)^{\frac{d(d-1)}{2}}$.

To prove the result, we thus only have to check that n and $\frac{d(d-1)}{2}$ have the same parity. When d is even, then $d = 2n$, and $\frac{d(d-1)}{2} = n(d-1)$ and n have the same parity. When d is odd, then $d-1 = 2n$, and $\frac{d(d-1)}{2} = nd$ and n also have the same parity. The result follows.

- Assume now Φ is of type E_6 . The character $\xi = \frac{\alpha_1^\vee - \alpha_3^\vee + \alpha_5^\vee - \alpha_6^\vee}{3}$ is then an element of $X_*(T)$, and if $t = \xi(c)$, we have $tCh(h) = Ch(h)$ for every $h \in H^1(\Gamma, K_{T \cap L_{E, \text{der}}, F})$, hence $\phi_D(tC) = \phi_D(C)$. On the other hand, since the group Y is of order 3 and χ is quadratic, it must be trivial, hence $\chi(\xi(c)) = 1$, and the result follows.
- Assume now Φ is of type E_7 . The group Y is then of order 2; moreover, the character $\xi = \frac{\alpha_2^\vee + \alpha_5^\vee + \alpha_7^\vee}{2}$ is an element of $X_*(T)$, and $t = \xi(c)$ is a representative of the nontrivial element of Y . By proposition 6.21, we have, for every h , $\xi(c)Ch(h) = Ch(e_4 e_6 e_7 h)$; hence, by proposition 6.22, $\phi_D(tC) = -\phi_D(C)$. On the other hand, since by [4, §1, proposition 29 (ii)], $\langle \rho, \alpha_i^\vee \rangle = 1$ for every i , we obtain $\langle 2\rho, \xi \rangle = 3$, hence $2\rho(t) = c^3$ and $\chi(t) = -1$, and the result follows.

□

Now we can define our linear form. For every $C \in Ch_{D,a}$, let O_C be the G_F -orbit of Ch_E containing C , and let R be a system of representatives of the

G_F -orbits in $Ch_{D,a}$. Set:

$$\lambda : f \in \mathcal{H}(X_E)^\infty \mapsto \sum_{C \in R} \sum_{C' \in O_C} f(C') \phi_D(C').$$

Since R is a finite set, proposition 7.1 implies that the double sum always converges.

Now that we have a linear form on $\mathcal{H}(X_E)^\infty$, we want to find a test vector for it. We start by the following propositions:

Proposition 7.23. *Let C be any element of Ch_E . There exists a unique element of Ch_D contained in the closure of $C \cup D$.*

Since C is a chamber, by [7, 2.4.4], the closure $cl(C \cup D)$ is a union of chambers of X_E . Hence D is contained in some chamber C' of that closure, which is then obviously an element of Ch_D .

On the other hand, let A_C be an apartment of X_E containing both C and D ; it then contains $cl(C \cup D)$. Consider the connected components of the complementary in $R(A_C)$ of the union of the walls containing $R(D)$; each one of them contains the geometric realization of a unique element of Ch_D . Let \mathcal{S} be the one containing $R(C)$; its closure contains $R(D)$, hence also the geometric realization of $cl(C \cup D)$, which proves the unicity of C' . \square

Proposition 7.24. *Let f_0 be a function on Ch_D satisfying the harmonicity condition, and let f be the function on Ch_E defined the following way: for every $C \in Ch_E$, if C_0 is the only element of Ch_D contained in the closure of $C \cup D$, $f(C) = (-q)^{-d(C, C_0)} f_0(C_0)$. Then $f \in \mathcal{H}(X_E)^\infty$.*

Let $K_{D,E}^0$ be the pro-unipotent radical of $K_{D,E}$; $K_{D,E}^0$ fixes every element of Ch_D pointwise. For every $C \in Ch_E$ and every $k \in K_{D,E}^0$, we then have:

$$f(kC) = (-q)^{-d(kC, kC_0)} f_0(kC_0) = (-q)^{-d(C, C_0)} f_0(C_0) = f(C);$$

since $K_{D,E}^0$ is an open compact subgroup of G_E , the smoothness of f is proved. Now we check the harmonicity condition. Let D' be any codimension 1 facet of X_E . Assume first that the closure of $D' \cup D$ contains at least one chamber C_1 of X_E ; it then contains exactly one element C_0 of Ch_D , namely the one whose geometric realization is contained in the same connected component as $R(C_1)$ of the complementary of the union of the walls containing $R(D)$ in any Γ -stable apartment containing $R(C_1)$; on the other hand, that closure also contains exactly one chamber C admitting D' as a wall. Set $\delta = d(C, C_0)$; if C' is any other chamber of X_E admitting D' as a wall, the closure of $C' \cap D$ contains D and C , hence contains also C_0 , and we have $d(C', C_0) = \delta + 1$. Since there are q such chambers, we obtain:

$$\sum_{C' \supset D'} f(C') = (-q)^{-\delta} f_0(C_0) + q(-q)^{-\delta-1} f_0(C_0) = 0.$$

Assume now that the closure of $D' \cup D$ does not contain any chamber. It then contains a unique facet D_0 of X_E of codimension 1 containing D ; moreover, if C is a chamber of X_E admitting D' as a wall, the only element C_0 of Ch_D contained in the closure of $C \cup D$ must admit D_0 as a wall. On the other hand, the group $K_{D' \cup D}$ permutes transitively the elements of Ch_D admitting D_0 as a wall; since there are $q + 1$ such chambers, and $q + 1$ chambers of X_E admitting D' as a wall as well, the restriction to the second ones of the application $C \mapsto C_0$ must be a bijection, and all of them are at the same distance δ from Ch_D . We then have:

$$\sum_{C \supset D'} f(C) = (-q)^{-\delta} \sum_{C_0 \supset D_0} f_0(C_0).$$

Since f_0 satisfies the harmonicity condition as a function on Ch_D , the right-hand side is zero, hence the left-hand side must be zero as well. Hence f satisfies the harmonicity condition and the proposition is proved. \square

Now let ϕ be the function on Ch_E derived from ϕ_D by the previous proposition. We say that ϕ is the *extension by harmonicity* of ϕ_D .

Proposition 7.25. *The function ϕ belongs to $\mathcal{H}(X_E)^\infty$, and is a test vector for λ .*

The fact that $\phi \in \mathcal{H}(X_E)^\infty$ is an immediate consequence of propositions 7.21 and 7.24. Now we prove that ϕ is a test vector for λ . First assume D is a single vertex x ; we then write Ch_x , $Ch_{x,a}$, ϕ_x instead of Ch_D , $Ch_{D,a}$, ϕ_D . We first prove the following lemma:

Lemma 7.26. *Let C be an element of Ch_a^0 such that $\phi(C) \neq 0$. Then $C \in Ch_{x,a}$.*

Assume $C \notin Ch_{x,a}$; there exists then another vertex x' of X_E whose geometric realization is in \mathcal{B}_F , belonging to C and such that $C \in Ch_{x',a}$. Let C_0 be the only element of Ch_x contained in the closure of $C \cup \{x\}$; the closure of C_0 must then contain a facet of dimension at least 1 of the closure of $\{x, x'\}$, whose geometric realization is contained in \mathcal{B}_F . Hence C_0 cannot belong to $Ch_{x,a}$, which implies that $\phi(C_0)$ must be zero, and $\phi(C)$ is then also zero by definition of ϕ . \square

According to this lemma, we have:

$$\lambda(\phi) = \sum_{C \in Ch_{x,a}} \phi_x(C) \phi(C) = \sum_{C \in Ch_{x,a}} 1 = \#(Ch_{x,a}).$$

Since $Ch_{x,a}$ is nonempty, $\lambda(\phi) \neq 0$ and the proposition is proved.

Now we deal with the cases where D is of nonzero dimension. As before, we denote by Φ_D the root system of \mathbb{G}_D , which is also the Levi subsystem of Φ generated by Σ_a , or equivalently the set of elements of Φ which are linear combinations with coefficients in \mathbb{Q} of the elements of Σ_a .

Remember that Ch_a^0 is the set of chambers of anisotropy class Σ_a containing a Γ -fixed facet of the same dimension as D .

Lemma 7.27. *Let C be an element of Ch_a^0 such that $\phi_D(C) \neq 0$, and let D' be the Γ -fixed facet of C of maximal dimension. There exists a Γ -stable apartment A of X_E containing both D and C , hence also D' , and D and D' are then facets of maximal dimension of A^Γ .*

Let C_0 be the only element of Ch_D contained in the closure of $C \cup D$; by definition of ϕ_D , we must have $C_0 \in Ch_{D,a}$, which implies that the intersection of C_0 and $\gamma(C_0)$ is D . Moreover, C is also the only element of $Ch_{D'}$ contained in the closure of $C_0 \cup D'$, hence $\gamma(C)$ is the only element of $Ch_{D'}$ contained in the closure of $\gamma(C_0 \cup D') = \gamma(C_0) \cup D'$.

Consider now the closure of $\gamma(C_0) \cup C$; it contains both $\gamma(C_0) \cup D'$ and $C \cup D$, and by the previous remarks it must contain C_0 and $\gamma(C)$ as well, hence also the closure of $C_0 \cup \gamma(C)$; by symmetry, these two closures are then equal. We have thus obtained a Γ -stable subset of X_E which is the closure of the union of two facets; by [7, proposition 2.3.1], that set is contained in some apartment A' of X_E , and by the same inductive reasoning as in proposition 4.1, we obtain a Γ -stable apartment A containing it, which must then satisfy the required conditions. \square

Let A be a Γ -stable apartment of X_E containing at least one chamber belonging to Ch_D , and let D', D'' be facets of maximal dimension of A^Γ . We denote by $d_\Gamma(D', D'')$ the combinatorial distance between D' and D'' inside the subcomplex A^Γ of X_E .

Lemma 7.28. *Let D', D'' be two distinct facets of maximal dimension of A^Γ , and let C' be a chamber of A containing D' .*

- *The parahoric subgroups $K_{D',E}$ and $K_{D'',E}$ of G_E fixing respectively D' and D'' are strongly associated (in the sense of [11, definition 3.1.1]).*
- *There exists a unique chamber C'' of A containing D'' and such that no minimal gallery between C' and C'' contains any other chamber containing either D' or D'' .*

Since D' and D'' both generate A^Γ as an affine subcomplex of X_E , the finite reductive groups $K_{D',E}/K_{D',E}^0$ and $K_{D'',E}/K_{D'',E}^0$ are both canonically isomorphic to $K_{A^\Gamma,E}/K_{A^\Gamma,E}^0$, and we have:

$$K_{D',E} = K_{A^\Gamma,E} K_{D',E}^0,$$

from which we deduce:

$$(K_{D',E} \cap K_{D'',E}) K_{D',E}^0 \supset K_{A^\Gamma,E} K_{D',E}^0 = K_{D',E}.$$

The other inclusion being obvious, we obtain in fact an equality. By switching D' and D'' in the previous reasoning, we also obtain:

$$(K_{D',E} \cap K_{D'',E}) K_{D'',E}^0 = K_{D'',E}.$$

Hence $K_{D',E}$ and $K_{D'',E}$ are strongly associated, as desired. It implies in particular that $K_{D',E}/K_{D',E}^0$ and $K_{D'',E}/K_{D'',E}^0$ are canonically isomorphic to each other.

Now we prove the second assertion. We first observe that the image of $K_{C,E} \subset K_{D',E}$ in $K_{D',E}/K_{D',E}^0$ is a Borel subgroup of $K_{D',E}/K_{D',E}^0$. Let now C'' be the chamber of A containing D'' and such that the image of $K_{C'',E}$ in $K_{D'',E}/K_{D'',E}^0$ is (up to the aforementioned canonical isomorphism) that same Borel subgroup. Assume there exists a minimal gallery $(C'_0 = C', C'_1, \dots, C'_r = C'')$ between C' and C'' such that C'_i contains either D' or D'' , say for example D' , for some $i \in \{1, \dots, r-1\}$. Then $K_{C'_i,E}$ is contained in $K_{D',E}$, and its image in $K_{D',E}/K_{D',E}^0$ is a Borel subgroup which must be different from $K_{C',E}/K_{D',E}^0$ since $C'_i \neq C'$; hence C' and C'_i are separated by at least one hyperplane H of A containing D' . Such a hyperplane must then contain the whole subcomplex A^Γ , and in particular D'' , and since H then also separates C'_i from C'' , the gallery has to cross it at least twice, which contradicts its minimality.

Now let C''' be another chamber satisfying the conditions of the second assertion. Since D' and D'' are distinct, we must have $C''' \neq C'$. On the other hand, let H be an hyperplane separating C'' from C''' . Since both C'' and C''' contain D'' , H must contain D'' as well, hence $K_{C''',E}/K_{D'',E}^0$ is a Borel subgroup of $K_{D'',E}/K_{D'',E}^0$ which is different from $K_{C'',E}/K_{D'',E}^0 \simeq K_{C',E}/K_{D',E}^0$; we deduce from this that there must exist a minimal gallery between C' and C''' containing C'' , and this is possible only if $C''' = C''$. The lemma is now proved. \square

Lemma 7.29. *Let D', D'' be two facets of maximal dimension of A^Γ , let C' be a chamber of A containing D' and let C'' be the only chamber of A containing D'' and contained in the closure of $C' \cup D''$. Then $\frac{d(C', C'')}{d_\Gamma(D', D'')}$ is a positive integer r_1 which does not depend on the choice of D', D'' and C' .*

It is easy to prove (by for example [5, lemma 4.2] and an obvious induction) that $q^{d(C', C'')} = [K_{C',E} : K_{C' \cup C'',E}] = [K_{C',E} : K_{C',E} \cap K_{C'',E}]$; moreover, we deduce immediately from the first assertion of lemma 7.28 that $[K_{C',E} : K_{C',E} \cap K_{C'',E}] = [K_{D',E} : K_{D',E} \cap K_{D'',E}]$. We thus only have to relate that last quantity to $d_\Gamma(D', D'')$.

As usual, we can without loss of generality assume that A^Γ is contained in $A_{0,E}$. Assume first D' and D'' are adjacent. Let $\Phi_{D'}$ be the Levi subsystem of Φ corresponding to the root system of $K_{D',E}/K_{D',E}^0$, which we can without loss of generality assume to be standard, and let α be any positive element of Φ corresponding to an hyperplane of \mathcal{A}_0 separating D' from D'' ; the set of such hyperplanes is then precisely the set of elements of Φ^+ contained in $\alpha + X_{D'}$, where $X_{D'}$ is the subgroup of $X^*(T_0)$ generated by $\Phi_{D'}$. We thus only have to check that the cardinality of $\Phi_{D',D''} = \Phi \cap (\alpha + X_{D'})$ is always the same.

- When Φ is of type A_{2n-1} , the simple roots contained in $\Phi_{D'}$ are the α_i with i odd. We then have $\Phi_{D',D''} = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for some i , and in particular $\Phi_{D',D''}$ always has 4 elements.
- When Φ is of type D_{2n+1} , every simple root in Φ^+ except α_1 is contained in $\Phi_{D'}$. The set $\Phi_{D',D''}$ is then the full set of the elements of Φ^+ which do

not belong to $\Phi_{D'}$; there are $4n$ such roots, which are precisely the roots of the form $\varepsilon_1 \pm \varepsilon_i$, $2 \leq i \leq 2n + 1$.

- When Φ is of type E_6 , the simple roots contained in $\Phi_{D'}$ are the α_i with $2 \leq i \leq 5$. The set $\Phi_{D',D''}$ then contains every positive element of the Levi subsystem of Φ generated by $\Phi_{D'}$ and α_j , with j being either 1 or 6, which do not belong to $\Phi_{D'}$. Since in both cases this Levi subsystem is of type D_5 , we are reduced to the previous case with $n = 2$, and we obtain in particular that the cardinality of $\Phi_{D',D''}$ is always 8.

In all these cases, the cardinality of $\Phi_{D',D''}$ is an integer r_1 which does not depend on the choice of D' and D'' .

Now we prove the general case by induction on $d_\Gamma(D', D'')$. Assume $d_\Gamma(D', D'') > 1$ and let D''' be a facet of maximal dimension of A^Γ distinct from D' and D'' and such that $d_\Gamma(D', D'') = d_\Gamma(D', D''') + d_\Gamma(D''', D'')$; D''' is then contained in the closure of $D' \cap D''$, hence also in the closure of $C' \cap C''$, and that closure must then contain an element C''' of $Ch_{D'''}$, which implies that $d(C', C'') = d(C', C''') + d(C''', C'')$. By induction hypothesis we have $d(C', C''') = r_1 d_\Gamma(D', D''')$ and $d(C''', C'') = r_1 d_\Gamma(D''', D'')$, hence $d(C', C'') = r_1 d_\Gamma(D', D'')$ and the lemma is proved. \square

Lemma 7.30. *Let D' be a facet of maximal dimension of A^Γ . There exists an integer r_2 such that for every facet of maximal dimension D'' of A^Γ , the number of $K_{D',F}$ -conjugates of D'' is precisely $q^{r_2 d_\Gamma(D', D'')}$. Moreover, we have $r_2 < r_1$.*

The number of $K_{D',F}$ -conjugates of D'' is precisely equal to $[K_{D',F} : K_{D',F} \cap K_{D'',F}]$, which cannot be greater than $[K_{D',E} : K_{D',E} \cap K_{D'',E}] = q^{r_1 d_\Gamma(D', D'')}$. Hence we already know that if r_2 exists, then $r_2 \leq r_1$.

By the same induction as in lemma 7.29 we are reduced to the case where D' and D'' are adjacent. We define $\Phi_{D',D''}$ the same way as in that lemma. Let $f_{D'}$ be the concave function on Φ associated with D' ; $K_{D',F}/(K_{D',F} \cap K_{D'',F})$ is then generated by the images of the root subgroups of $K_{D',F}$ corresponding to elements α of $\Phi_{D',D''}$ such that $f_{D'}(\alpha)$ is an integer, which by definition of C_0 and D' is true if and only if α is the sum of an even number of simple roots of Φ^+ . We thus only have to examine the different cases:

- when Φ is of type A_{2n-1} , $\Phi_{D',D''}$ always contains two such elements (either $\alpha_{2i-1} + \alpha_{2i}$, $\alpha_{2i} + \alpha_{2i+1}$ or $\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}$, depending on D');
- when Φ is of type D_{2n+1} , the elements of $\Phi_{D',D''}$ satisfying that condition are the $\varepsilon_1 \pm \varepsilon_i$ with i being of some given parity (which depends on D'), and there are $2n$ such roots;
- when Φ is of type E_6 , we are once again reduced to the case D_5 and $\Phi_{D',D''}$ then contains 4 elements satisfying the required condition.

Hence in all these cases, r_2 exists and is strictly smaller than r_1 , as required. \square

Remark: in all cases, we have $r_1 = 2r_2$, which is a predictable result since the ramification index of $[E : F]$ is 2. We will not use this fact in the sequel, though.

Now we prove proposition 7.25. By lemma 7.29, for every D' and every $C' \in Ch_{D'}$, if C is the only element of Ch_D contained in the closure of $D \cup C'$, we have:

$$\phi_D(C') = (-q)^{-r_1 d_\Gamma(D, D')} \phi_D(C),$$

hence:

$$\sum_{C' \in Ch_{D'}} \phi_D(C') = (-q)^{-r_1 d_\Gamma(D, D')} \sum_{C \in Ch_D} \phi_D(C),$$

Let d' be the dimension of D ; we have the following lemma:

Lemma 7.31. *Let W' be the affine Weyl group of G relative to T_0 ; the sum of the Poincaré series for a group of type $A_{d'}$ is:*

$$\sum_{w \in W'} x^{l(w)} = \frac{1 - x^{d'+1}}{(1 - x)^{d'+1}}.$$

According to a formula given in the proof of [14, corollary 3.4], we have:

$$\sum_{w \in W'} x^{l(w)} = \prod_{i=1}^{d'} \frac{1 - x^{m_i+1}}{(1 - x)(1 - x)^{m_i}},$$

where $m_1, \dots, m_{d'}$ are the exponents of W' (see [3, §6.2]). On the other hand, according to [4, plate I (X)], we have $m_i = i$ for every i . The lemma follows then by an easy computation. \square

We see immediately from this lemma that as soon as $|x| < 1$, the sum in the left-hand side cannot be zero. Denote by $s(x)$ that sum.

By proposition 7.1, the sum:

$$\sum_{C \in Ch_a^0} |\phi_D(C)|$$

converges, and we obtain, using lemmas 7.29 and 7.30 and taking into account the fact that r_1 and r_2 happen to be always even:

$$\sum_{C \in Ch_a^0} \phi_D(C) = \#(Ch_{D,a}) s(q^{r_2 - r_1}).$$

Since $r_2 < r_1$, the right-hand side is obviously nonzero. The proposition is now proved. \square

7.4. An Iwahori-spherical test vector. In this last section, we prove that it is always possible to use a suitably chosen Iwahori-spherical vector as a test vector. We start by the following lemma:

Lemma 7.32. *For every codimension 1 facet D of X_E , we have:*

$$\sum_{C \supset D} \phi_C = 0.$$

Let D be such a facet, and let C' be any chamber of X_E : we have:

$$\sum_{C \supset D} \phi_C(C') = \sum_{C \supset D} (-q)^{-d(C, C')}.$$

Consider the closure $cl(D \cup C')$; by [7, I, proposition 2.3.1], it is contained in an apartment A of X_E , and even in one of the two half-apartments of A delimited by the wall containing D . Hence there exists exactly one chamber C'' containing D and contained in $cl(D \cup C')$. Set $\delta = d(C'', C')$; if C''' is another chamber of X_E containing D , the closure of $C''' \cup C'$ must then contain C'' , and since C''' is neighboring C'' , we must have $d(C''', C') = \delta + 1$. Hence we have:

$$\sum_{C \supset D} \phi_C(C') = (-q)^{-\delta} + q((-q)^{-\delta-1}) = 0.$$

The lemma is then proved. \square

Now we check that we can use some well-chosen Iwahori-spherical vector as a test vector when G is not of type A_{2n} . In the case of type A_{2n} , we already know by proposition 7.5 that it is true.

Proposition 7.33. *Assume G is not of type A_{2n} . Let λ be any nonzero element of $\mathcal{H}(X_E)^{G_F, \chi}$, viewed as a linear form on $\mathcal{H}(X_E)^\infty$. Let C_0 be any element of Ch_a^0 and let ϕ_{C_0} be the Iwahori-spherical vector associated to C_0 . Then ϕ_{C_0} is a test vector for λ .*

We use the same argument as in [10, proposition 6.2]: since St_E is an irreducible representation, it is generated by any of its nonzero vectors, for example an Iwahori-spherical vector ϕ . We deduce from this that $\mathcal{H}(X_E)^\infty$ is generated as a \mathbb{C} -vector space by the G_E -conjugates of ϕ , which are the Iwahori-spherical vectors ϕ_C attached to every chamber C of X_E .

By lemma 7.32, the Iwahori-spherical vectors satisfy relations between each other which are similar to the harmonicity condition. Let λ be a nonzero (G_F, χ) -equivariant linear form on $\mathcal{H}(X_E)^\infty$. Assume $\lambda(f_{C_0}) = 0$. Then we prove in a similar way as for elements of $\mathcal{H}(X_E)^{G_F, der}$, using corollary 5.16 and propositions 6.6, 6.11 and 6.19, that $\lambda(\phi_C) = 0$ for every $C \in Ch_E$ as well, which implies $\lambda = 0$, and we thus reach a contradiction. Hence ϕ_{C_0} is a test vector for λ and the corollary holds. \square

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